

Chapter 5: Killing and Consumption

Density dependence is typically due to another variable of the system

$$R_T = R + cN$$

$$\frac{dN}{dt} = \frac{bRN}{h + R} - \delta N = \frac{b(R_T - cN)N}{h + R_T - cN} - \delta N = bN \left(1 - \frac{h}{h + R_T - cN} \right) - \delta N$$

Closed system

Question 5.1

Bacteria in a chemostat: birth rate proportional to consumption aR

$$\frac{dR}{dt} = s - wR - aRN$$

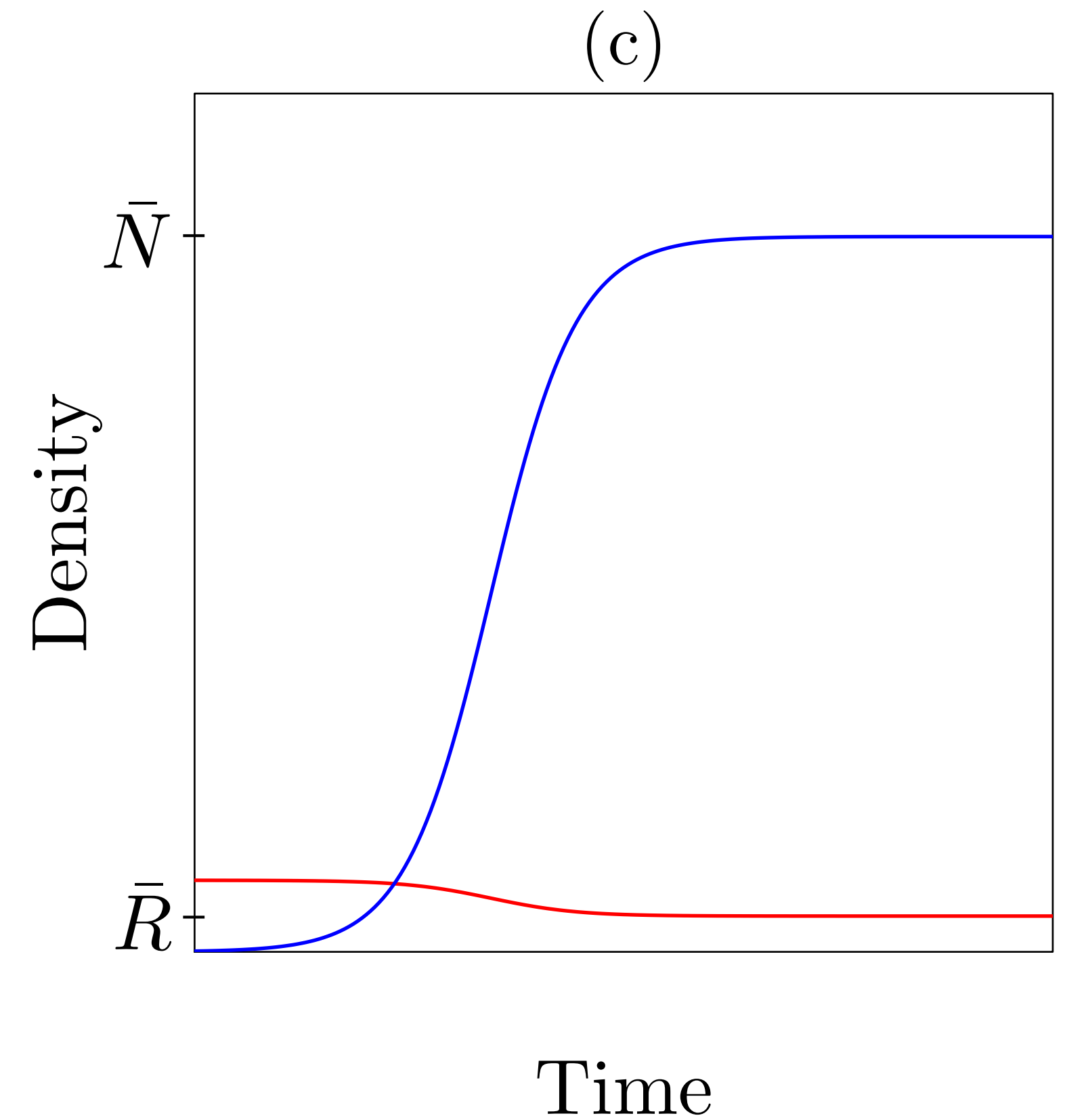
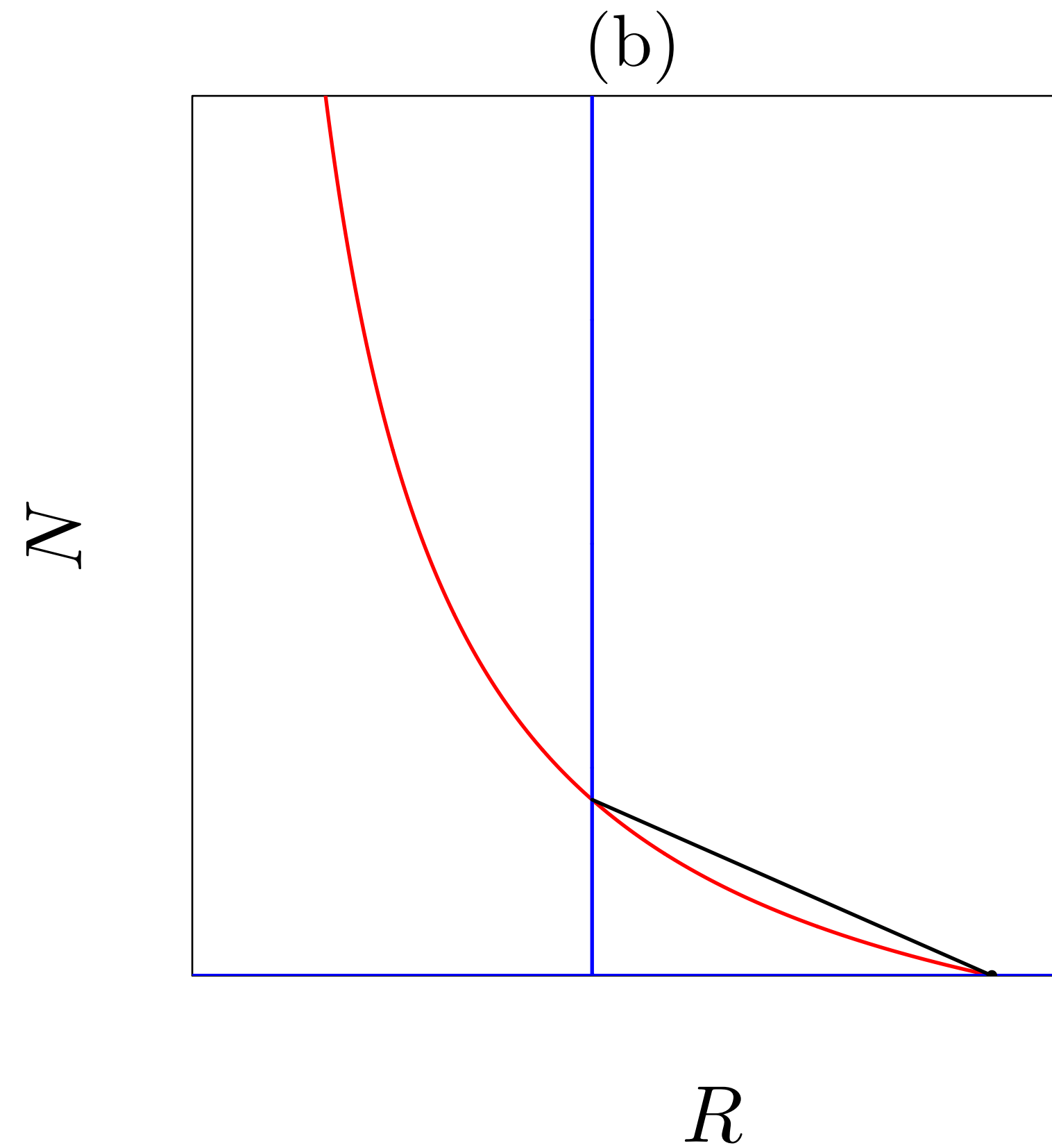
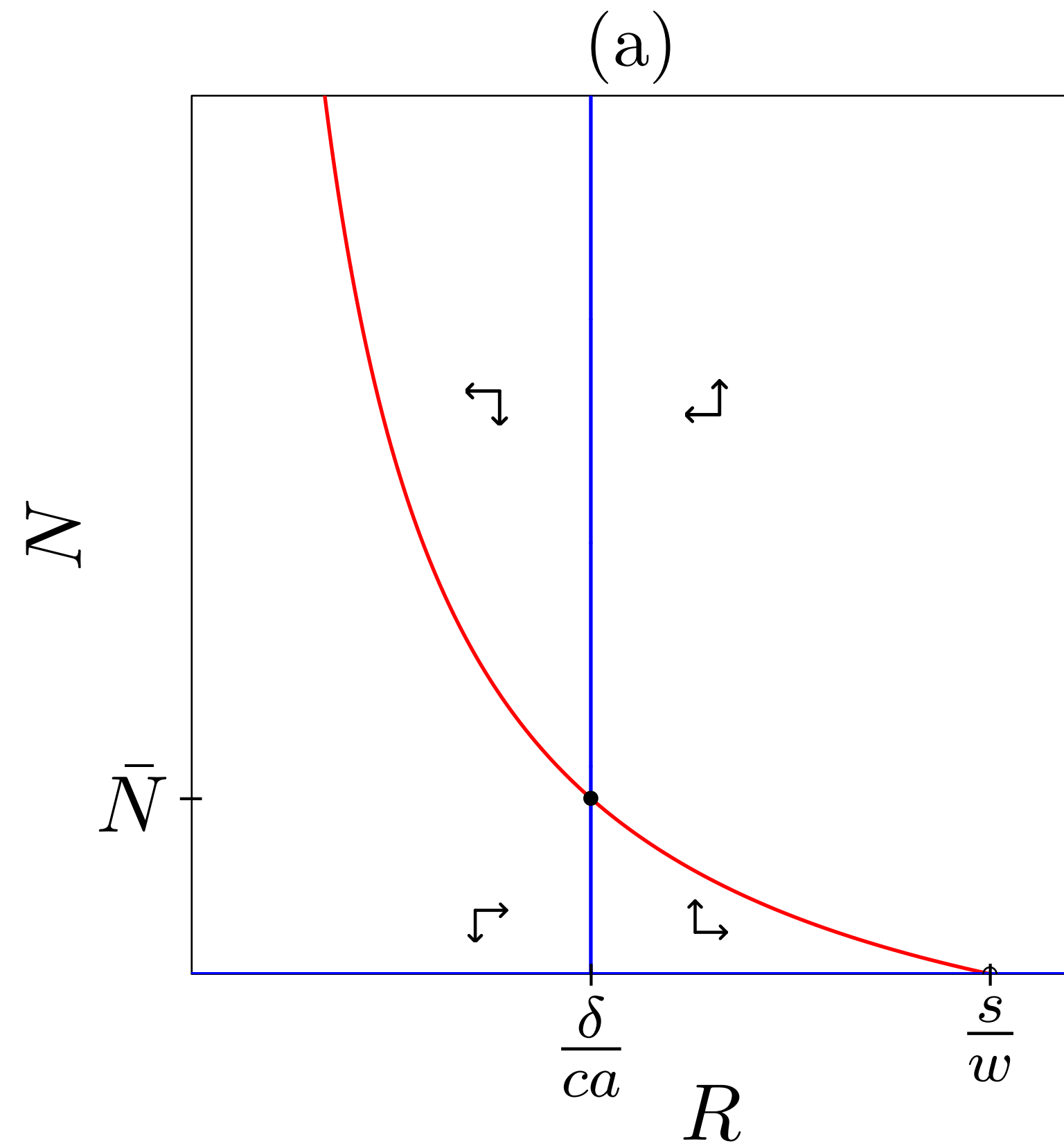
$$\frac{dN}{dt} = caRN - (w + d)N = caRN - \delta N$$

$$R = \frac{\delta}{ca}, \quad \bar{N} = \frac{sc}{\delta} - \frac{w}{a}$$

$$R_0 = \frac{ca\bar{R}}{\delta} = \frac{cas}{\delta w}$$

nullclines: $R' = 0$: $N = \frac{s}{aR} - \frac{w}{a}$ and: $N' = 0$: $N = 0$ or $R = \frac{\delta}{ca}$

Bacteria in a chemostat: birth rate proportional to consumption aR



$$\frac{dR}{dt} = s - wR - aRN$$

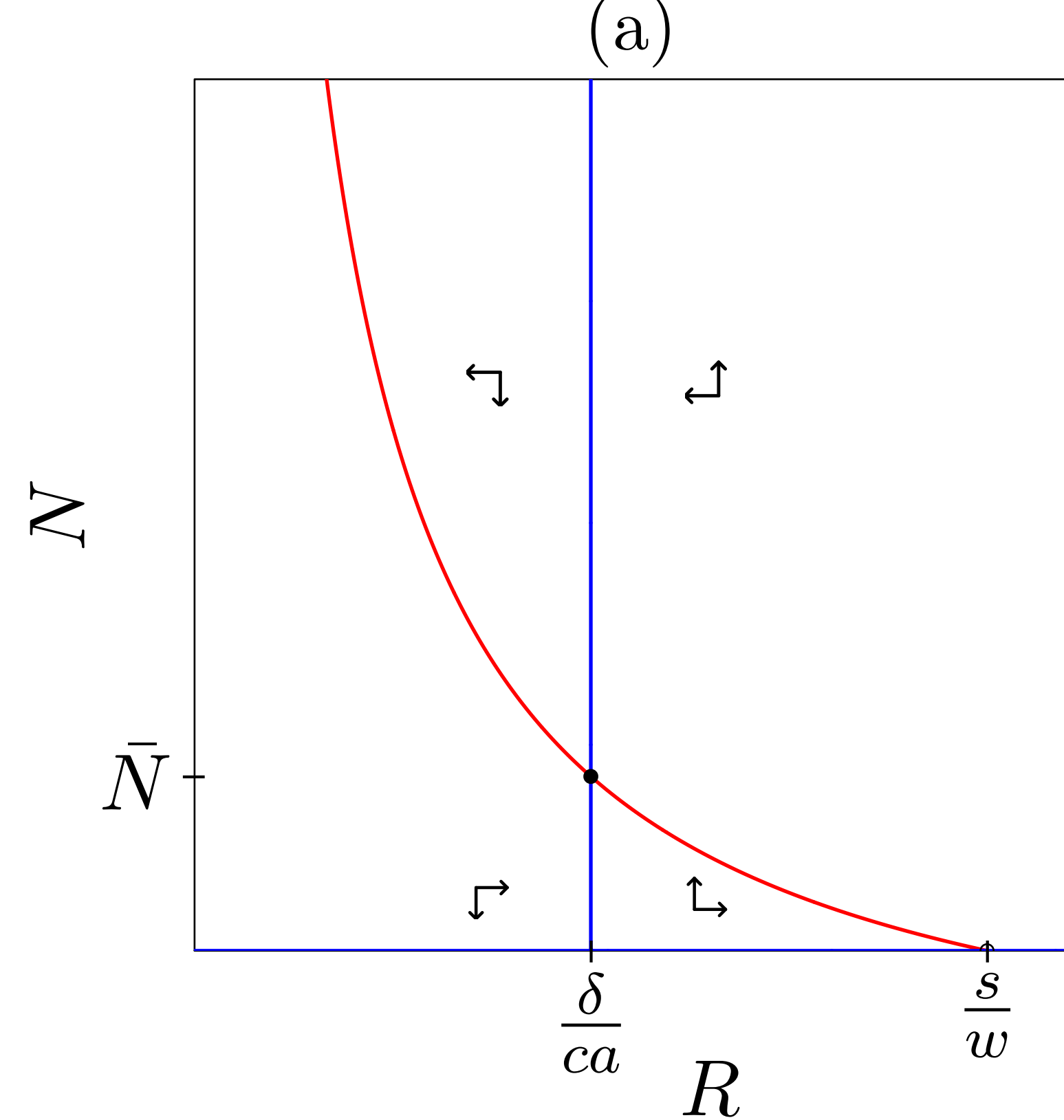
$$\frac{dN}{dt} = caRN - (w + d)N = caRN - \delta N$$

nullclines $N = \frac{s}{aR} - \frac{w}{a}$

$N = 0$ or $R = \frac{\delta}{ca}$

$$\frac{dR}{dt} = s - wR - aRN$$

$$\frac{dN}{dt} = caRN - (w + d)N = caRN - \delta N$$



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix}_{|(\bar{R}, \bar{N})} = \begin{pmatrix} -w - a\bar{N} & -a\bar{R} \\ ca\bar{N} & ca\bar{R} - \delta \end{pmatrix} = \begin{pmatrix} -w - a\bar{N} & -\delta/c \\ ca\bar{N} & 0 \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

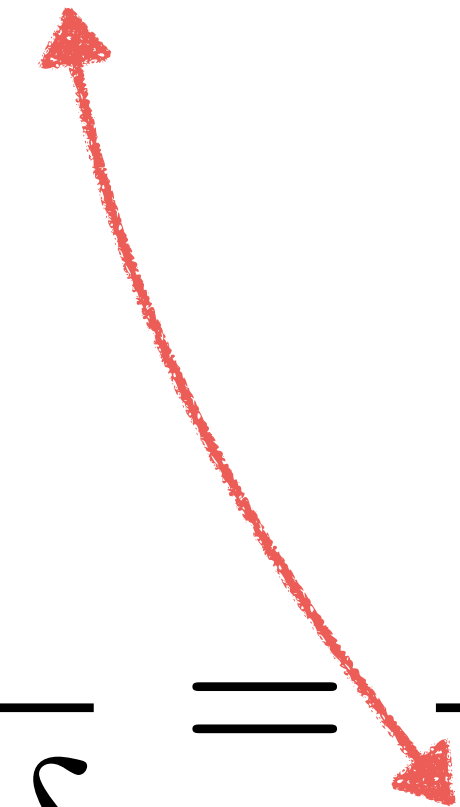
$$\text{tr} = -\alpha < 0.$$

$$\det = 0 - -\beta\gamma = \beta\gamma > 0$$

Saturated consumption in a chemostat, birth rate proportional to consumption

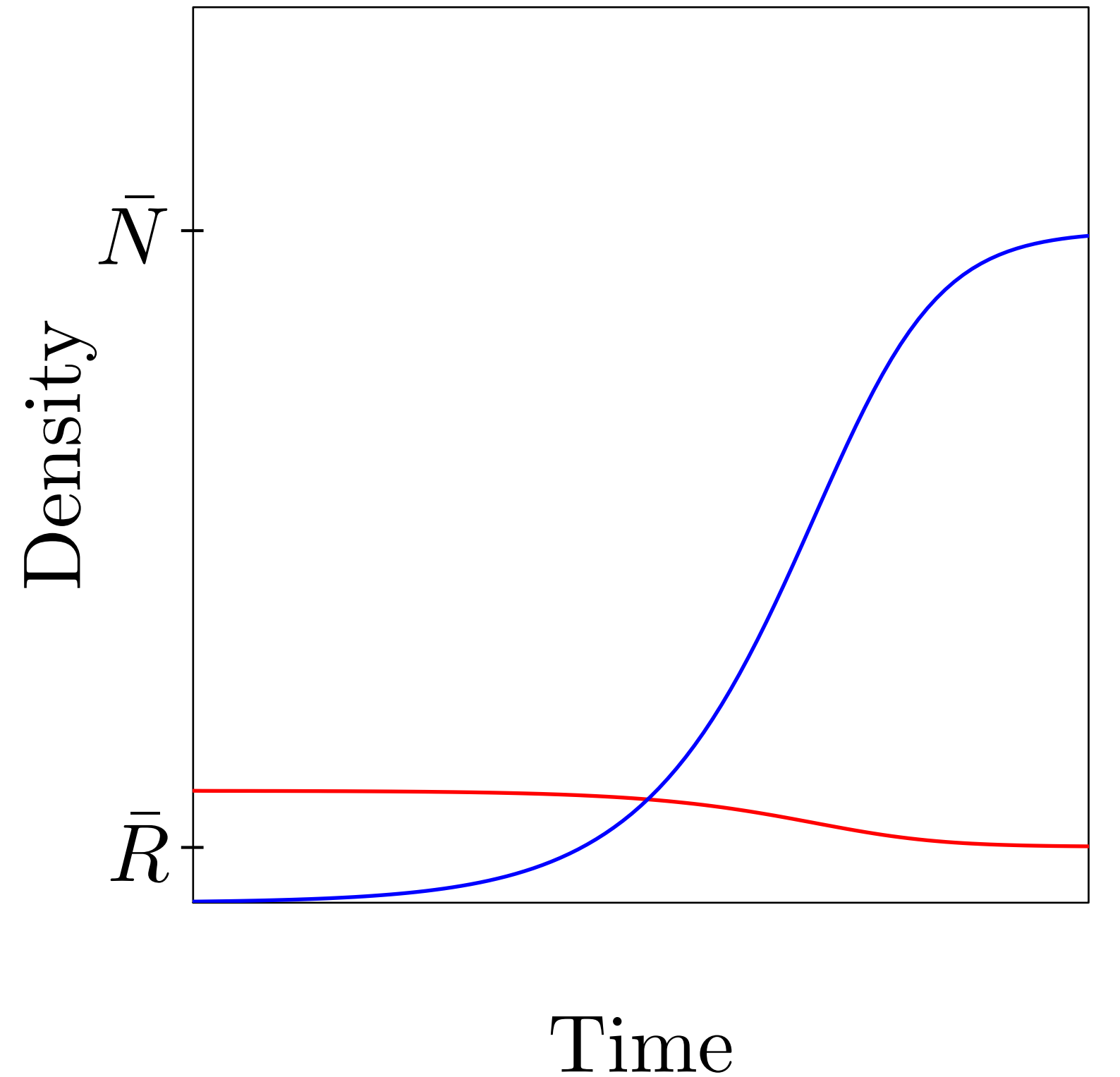
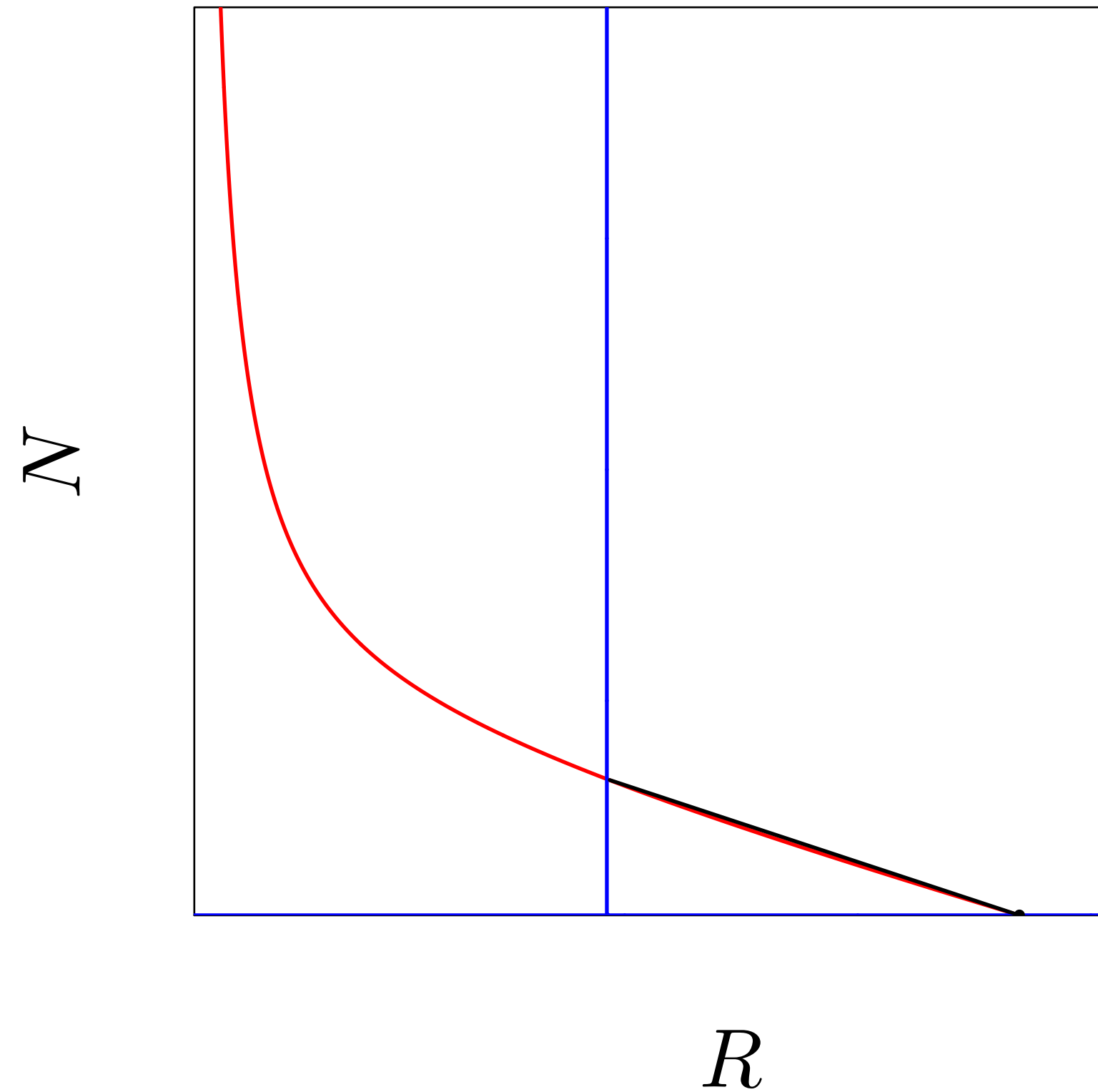
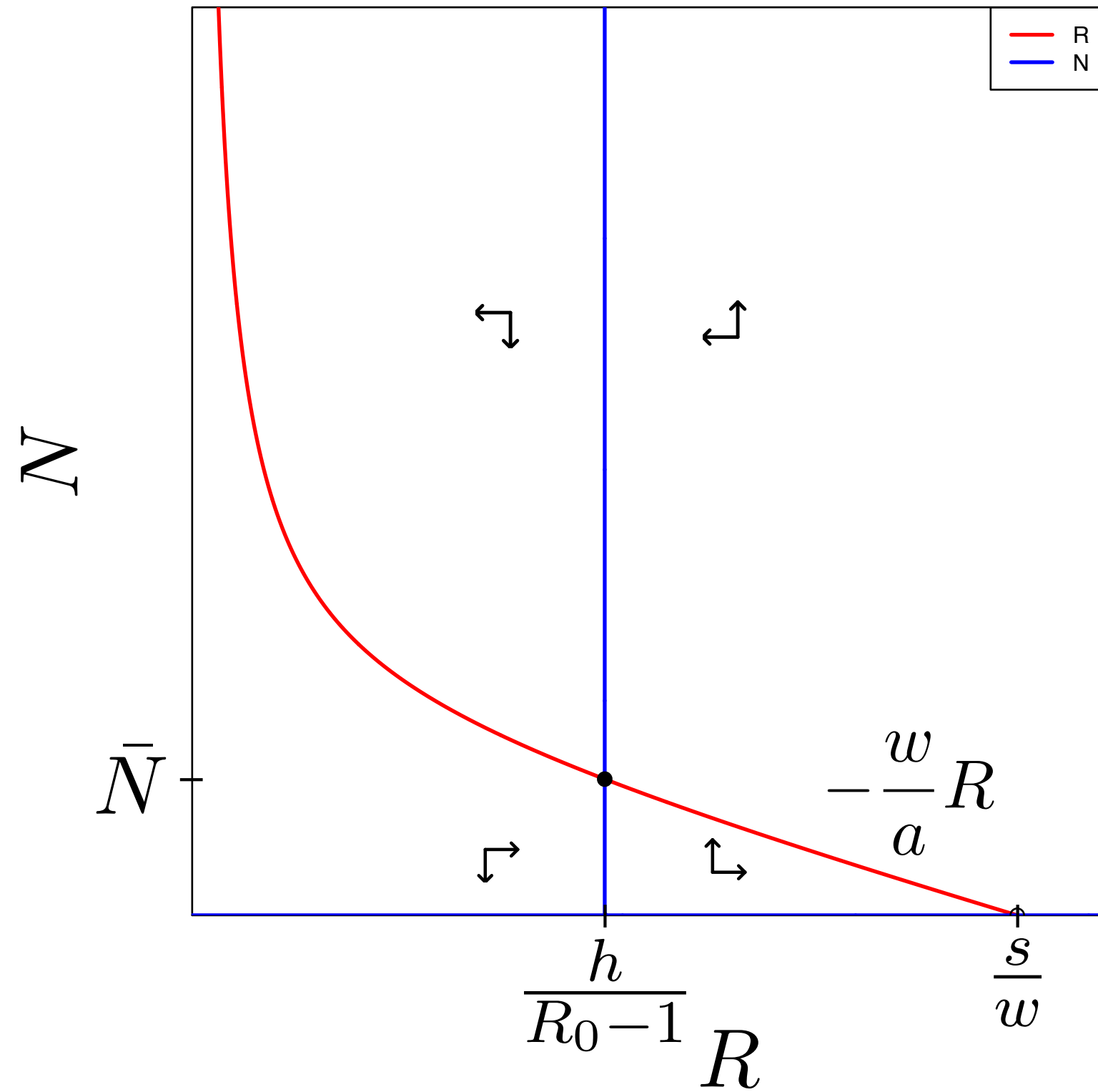
$$\frac{dR}{dt} = s - wR - \frac{aRN}{h + R} \quad \text{and} \quad \frac{dN}{dt} = \frac{caRN}{h + R} - (w + d)N = \frac{caRN}{h + R} - \delta N$$

$$R_0 = \frac{cas}{\delta(wh + s)} \quad \text{or} \quad R_0 = \frac{ca}{\delta}$$

$$\frac{dN}{dt} = 0 \quad \text{gives} \quad \bar{R} = \frac{h\delta}{ca - \delta} = \frac{h}{R_0 - 1}$$


$$\frac{dR}{dt} = s - wR - \frac{aRN}{h+R} \quad \text{and} \quad \frac{dN}{dt} = \frac{caRN}{h+R} - (w+d)N = \frac{caRN}{h+R} - \delta N$$

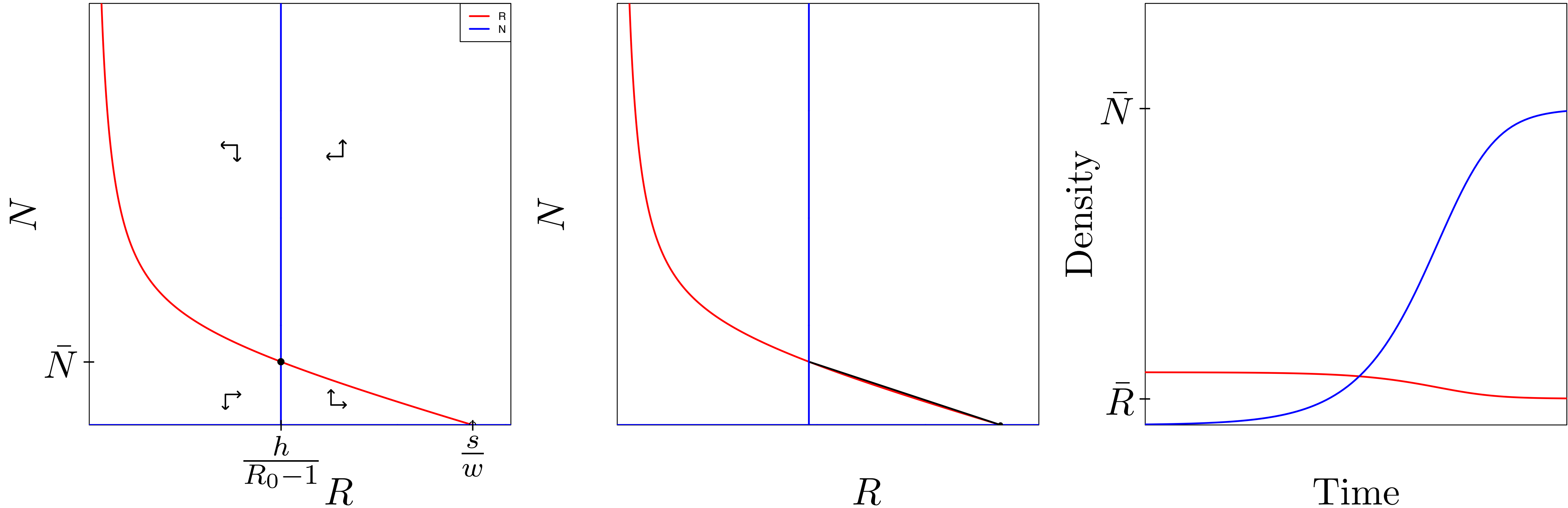
(a)
(b)
(c)



$$\frac{dR}{dt} = 0 \quad \text{gives} \quad s - wR = \frac{aRN}{h+R} \Leftrightarrow N = \frac{(h+R)(s-wR)}{aR} = \frac{h+R}{a} \left(\frac{s}{R} - w \right)$$

$$\frac{dR}{dt} = s - wR - \frac{aRN}{h+R} \quad \text{and} \quad \frac{dN}{dt} = \frac{caRN}{h+R} - (w+d)N = \frac{caRN}{h+R} - \delta N$$

(a)
(b)
(c)



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix}_{|(\bar{R}, \bar{N})} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$



Replicating resource: Lotka-Volterra model, birth rate proportional to consumption

$$\frac{dR}{dt} = rR(1 - R/K) - aRN \ , \quad \frac{dN}{dt} = caRN - \delta N \ .$$

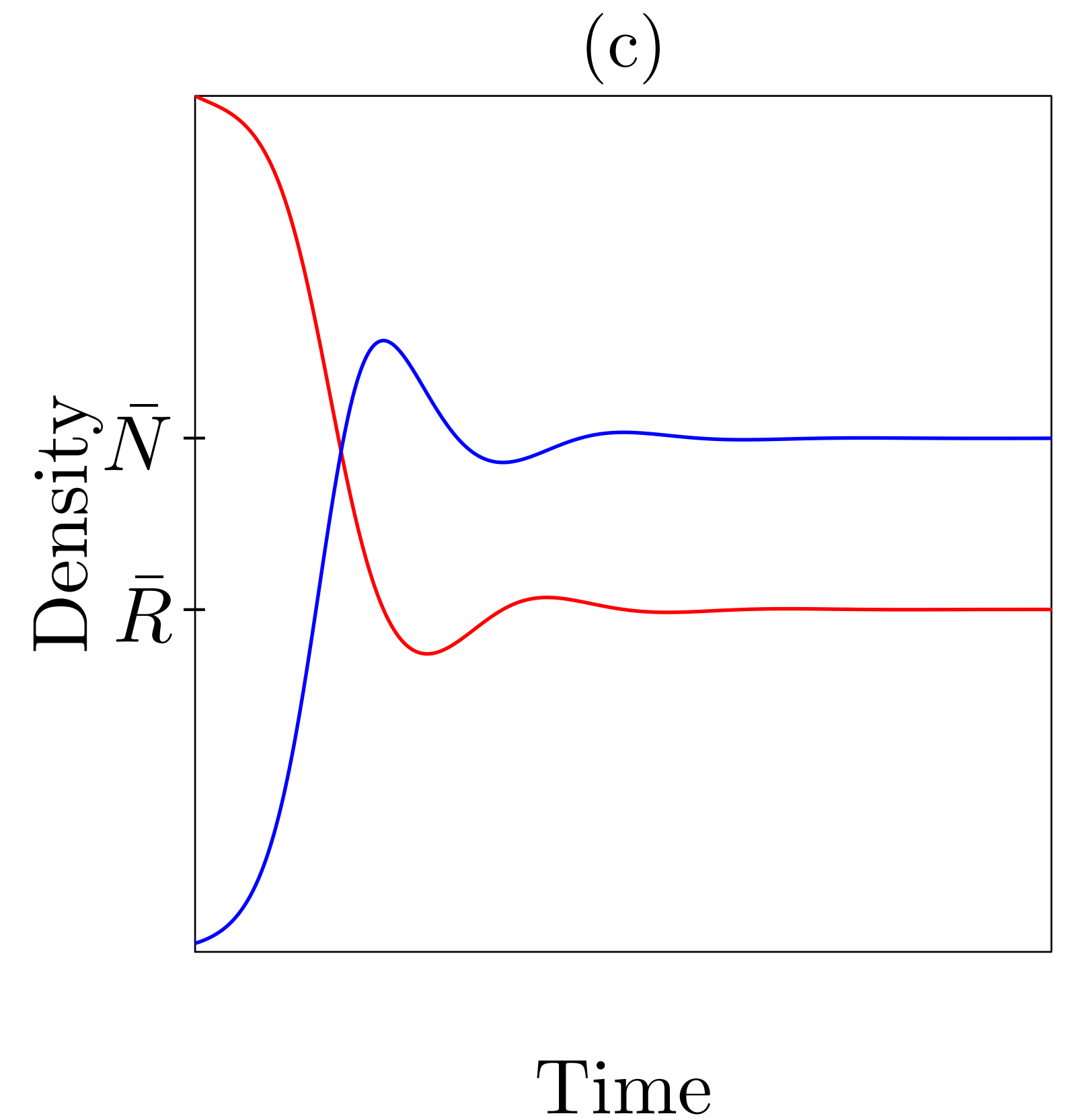
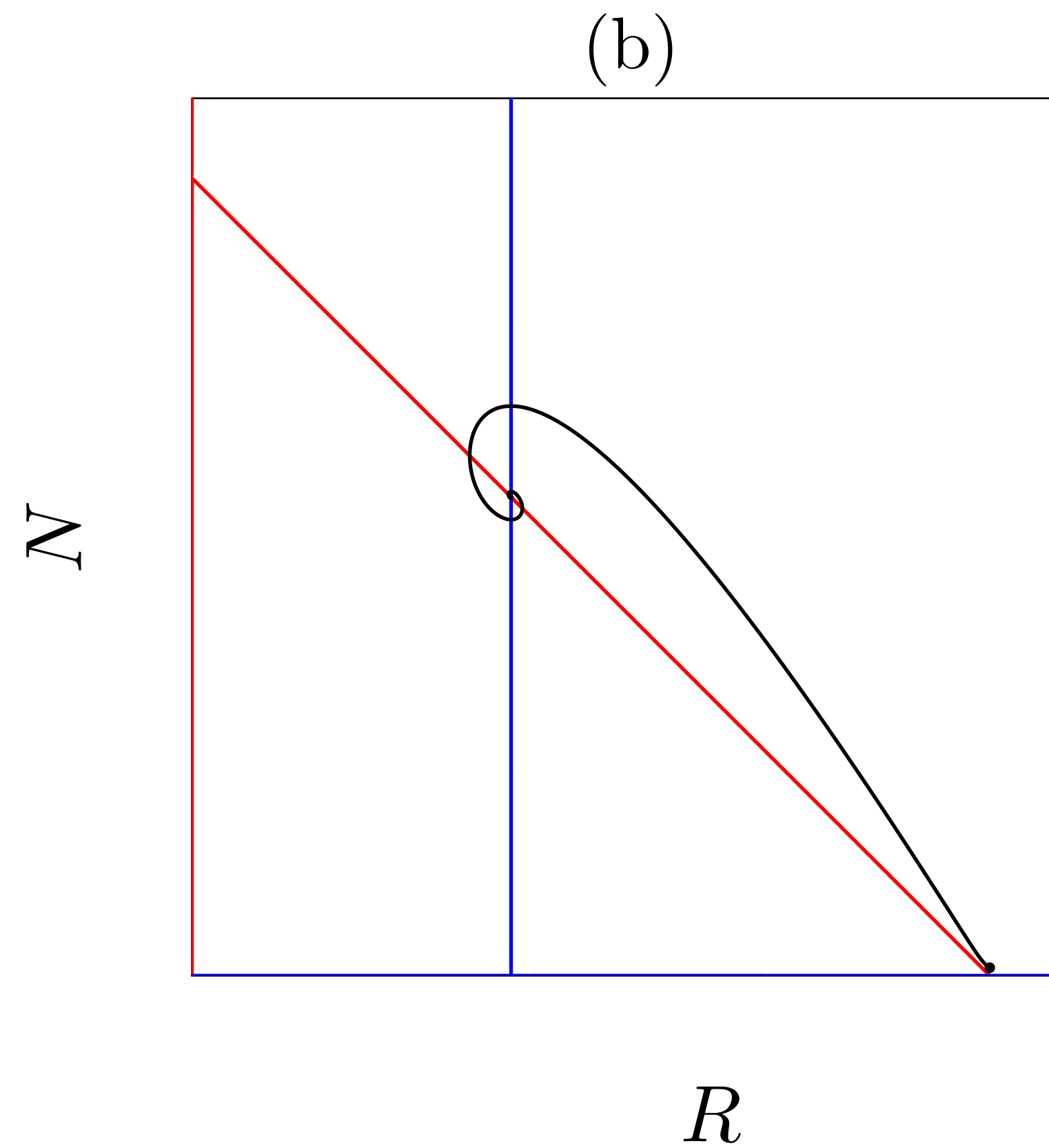
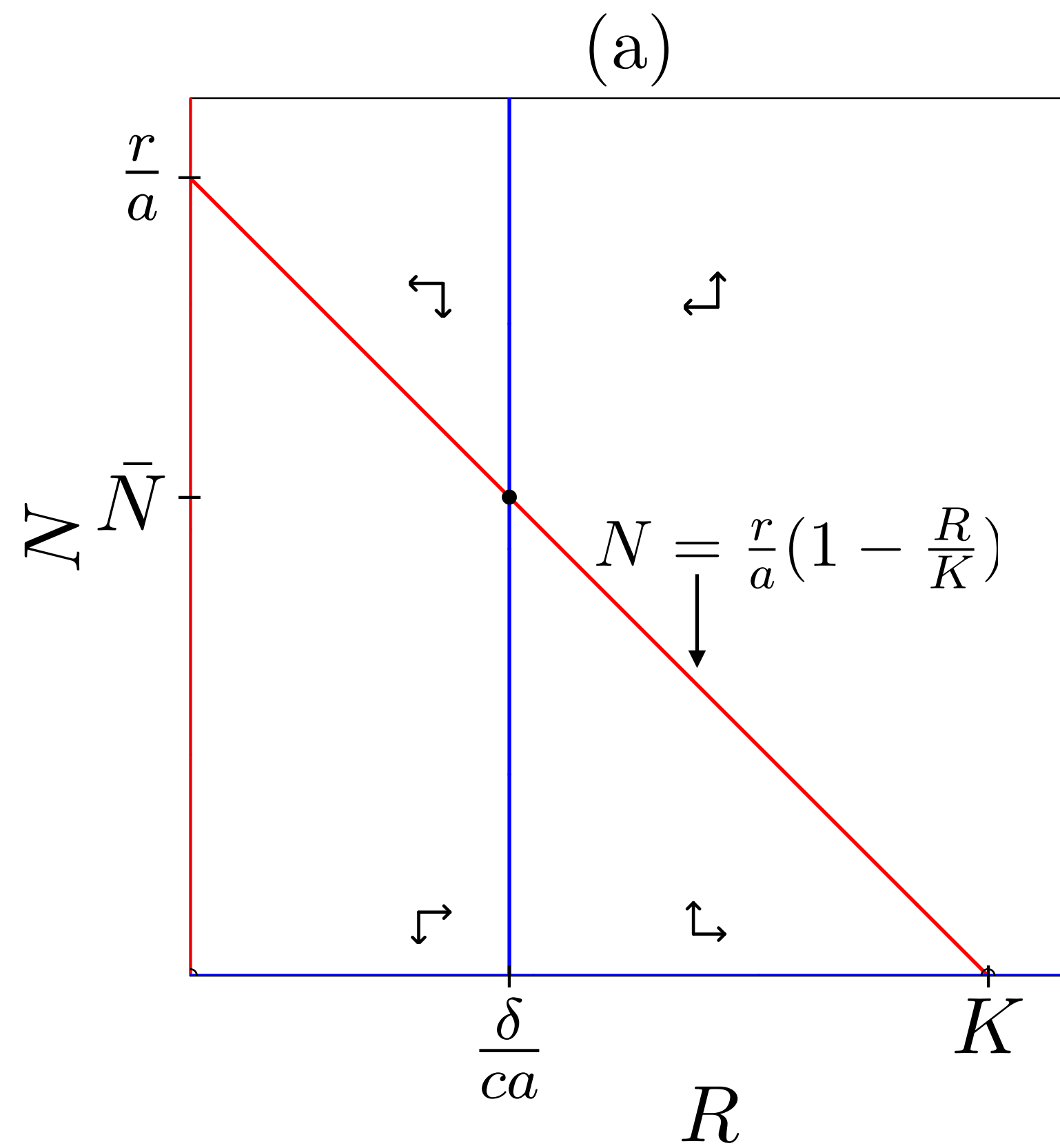
$$\frac{dN}{dt} = 0 \quad \text{gives} \quad \bar{R} = \frac{\delta}{ca}$$

$$\frac{dR}{dt} = 0 \quad \text{gives} \quad N = \frac{r}{a} \left(1 - \frac{R}{K}\right)$$

$$\text{Steady states: } (\bar{R}, \bar{N}) = (0, 0), (K, 0) \quad \text{and} \quad \left(\frac{\delta}{ca}, \frac{r}{a} \left[1 - \frac{\delta}{caK}\right]\right)$$

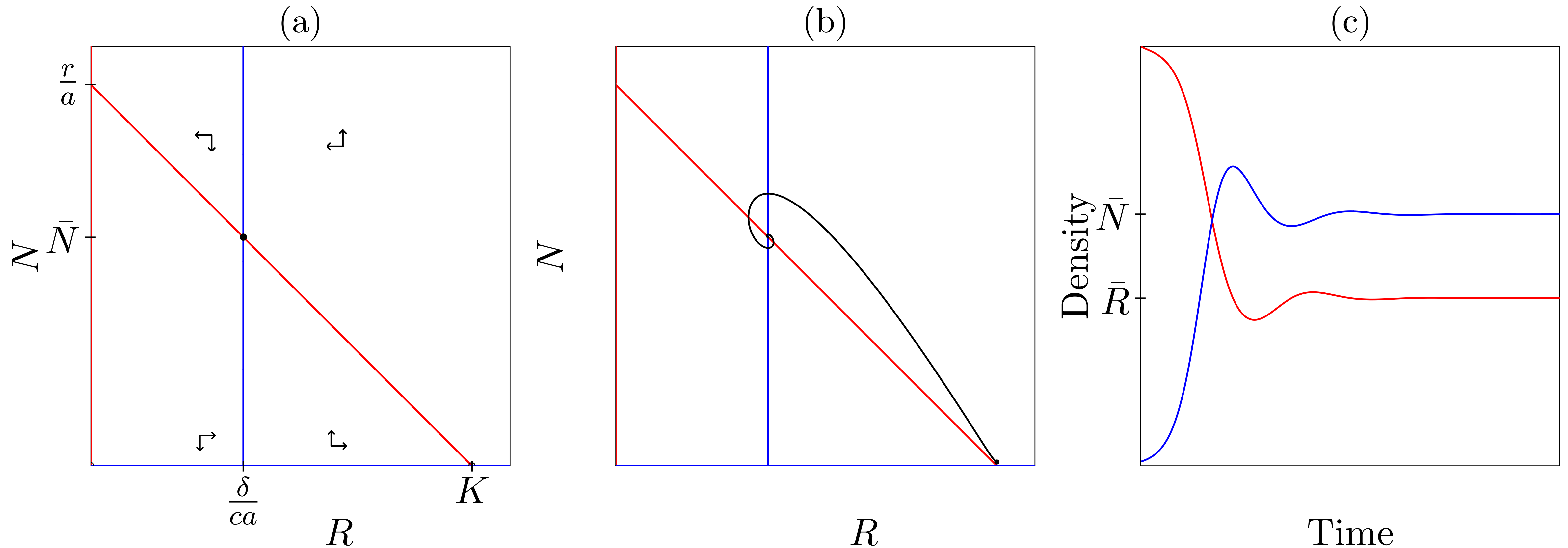
Nullclines Lotka-Volterra model

$$\frac{dR}{dt} = rR(1 - R/K) - aRN, \quad \frac{dN}{dt} = caRN - \delta N.$$



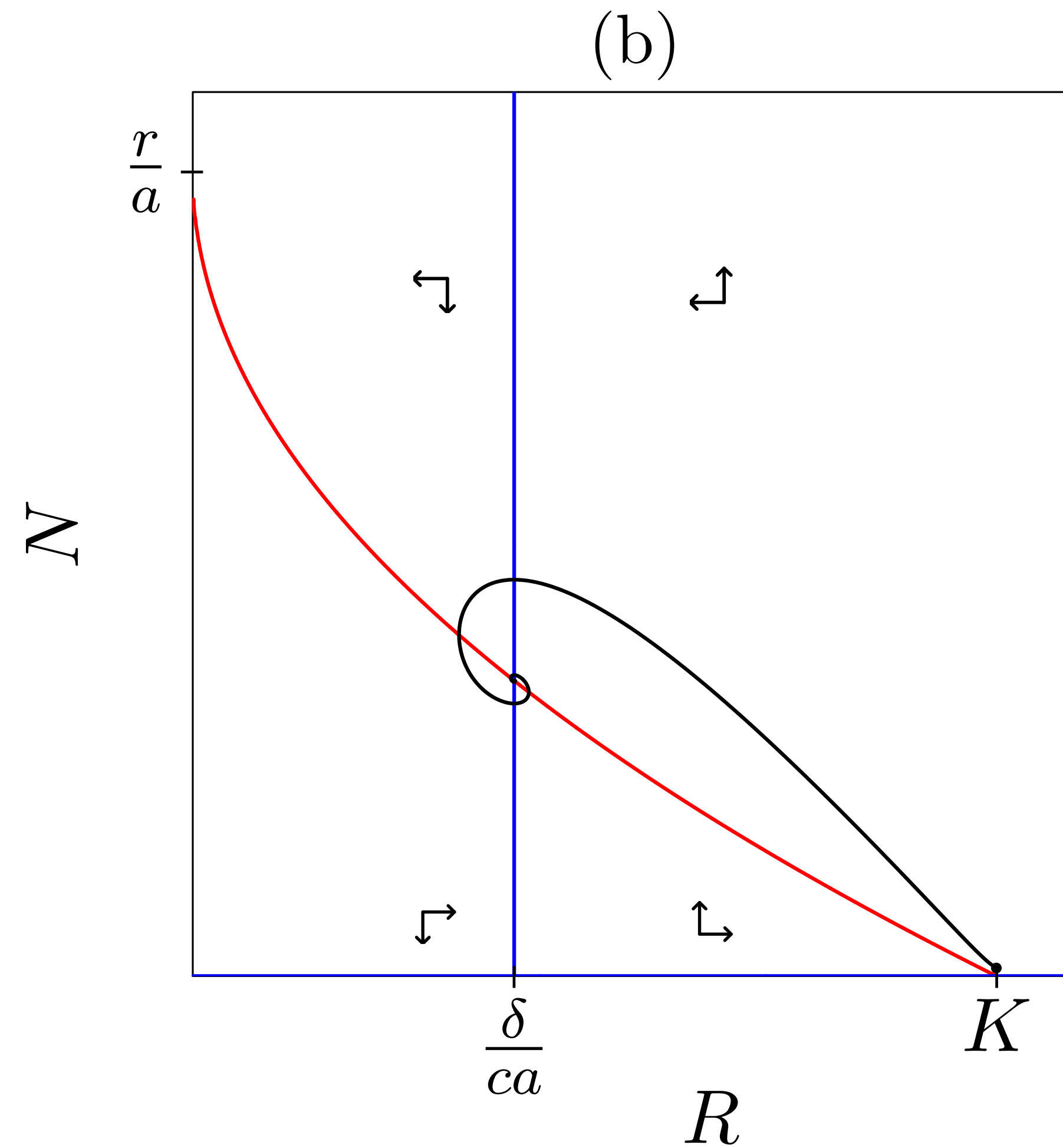
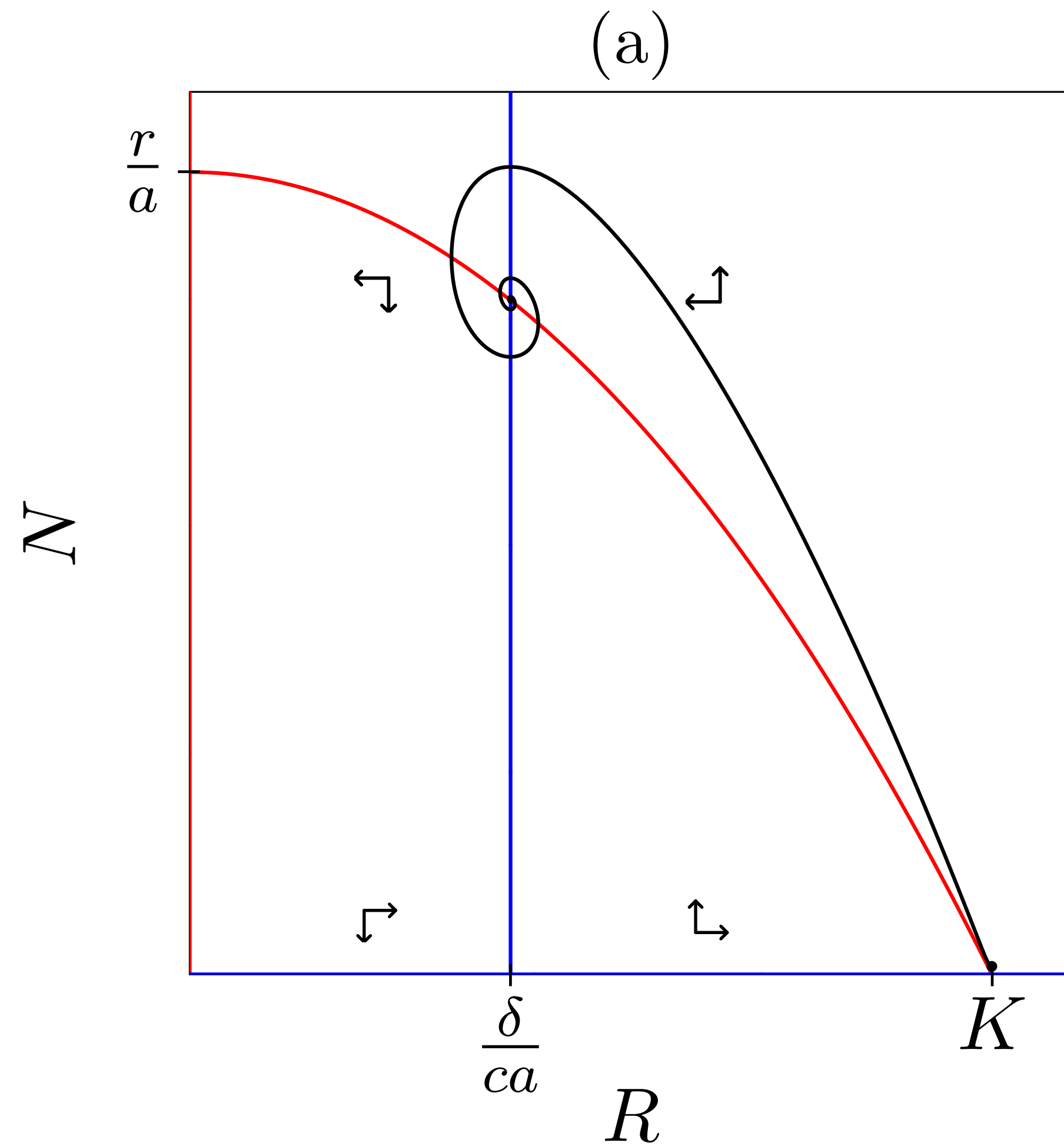
Jacobian Lotka-Volterra model

$$\frac{dR}{dt} = rR(1 - R/K) - aRN \quad , \quad \frac{dN}{dt} = caRN - \delta N \quad .$$



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix}_{|(\bar{R}, \bar{N})} = \begin{pmatrix} r - \frac{2r}{K} \bar{R} - a\bar{N} & -a\bar{R} \\ ca\bar{N} & ca\bar{R} - \delta \end{pmatrix} = \begin{pmatrix} -\frac{r\delta}{caK} & -\delta/c \\ ca\bar{N} & 0 \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

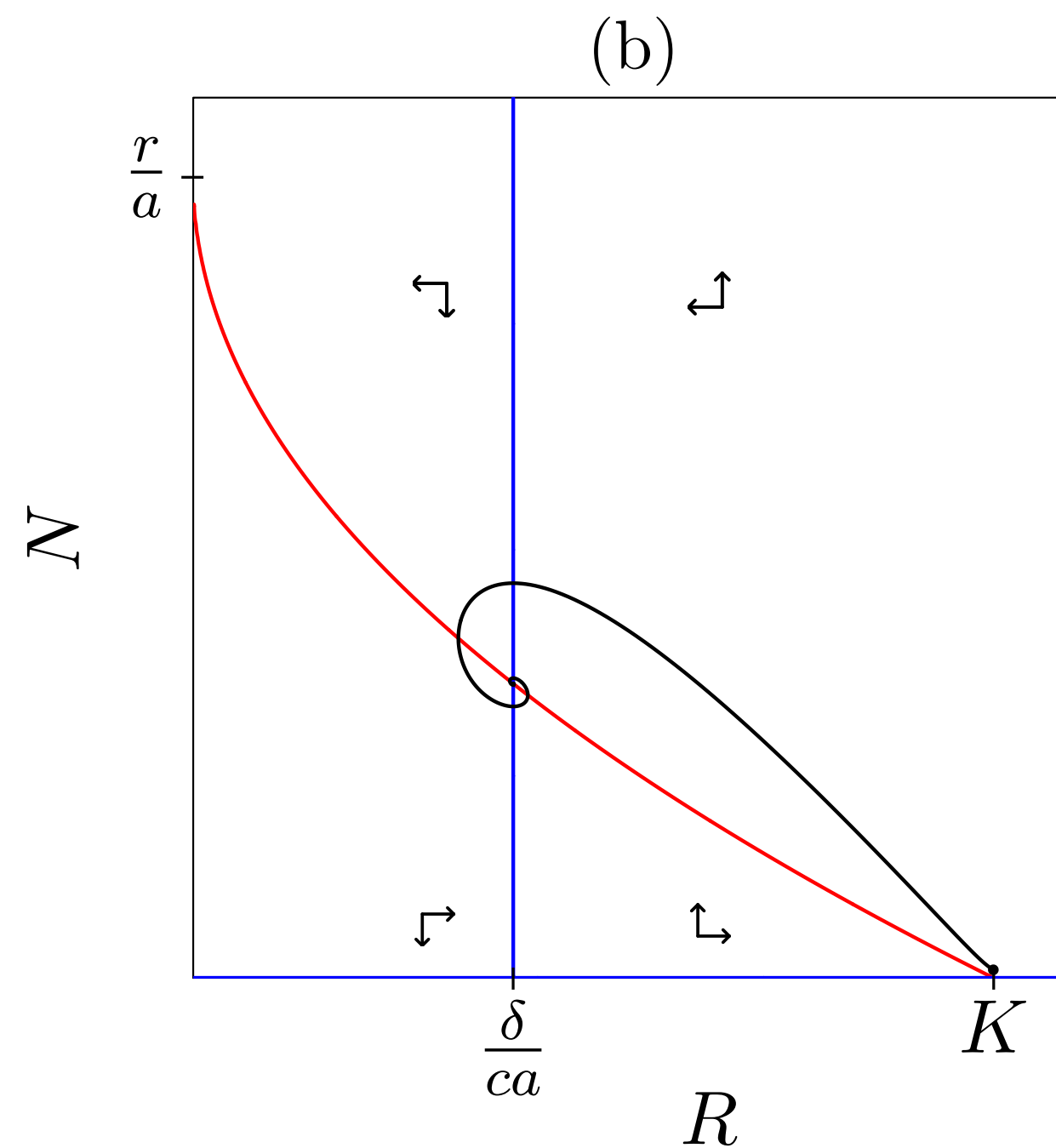
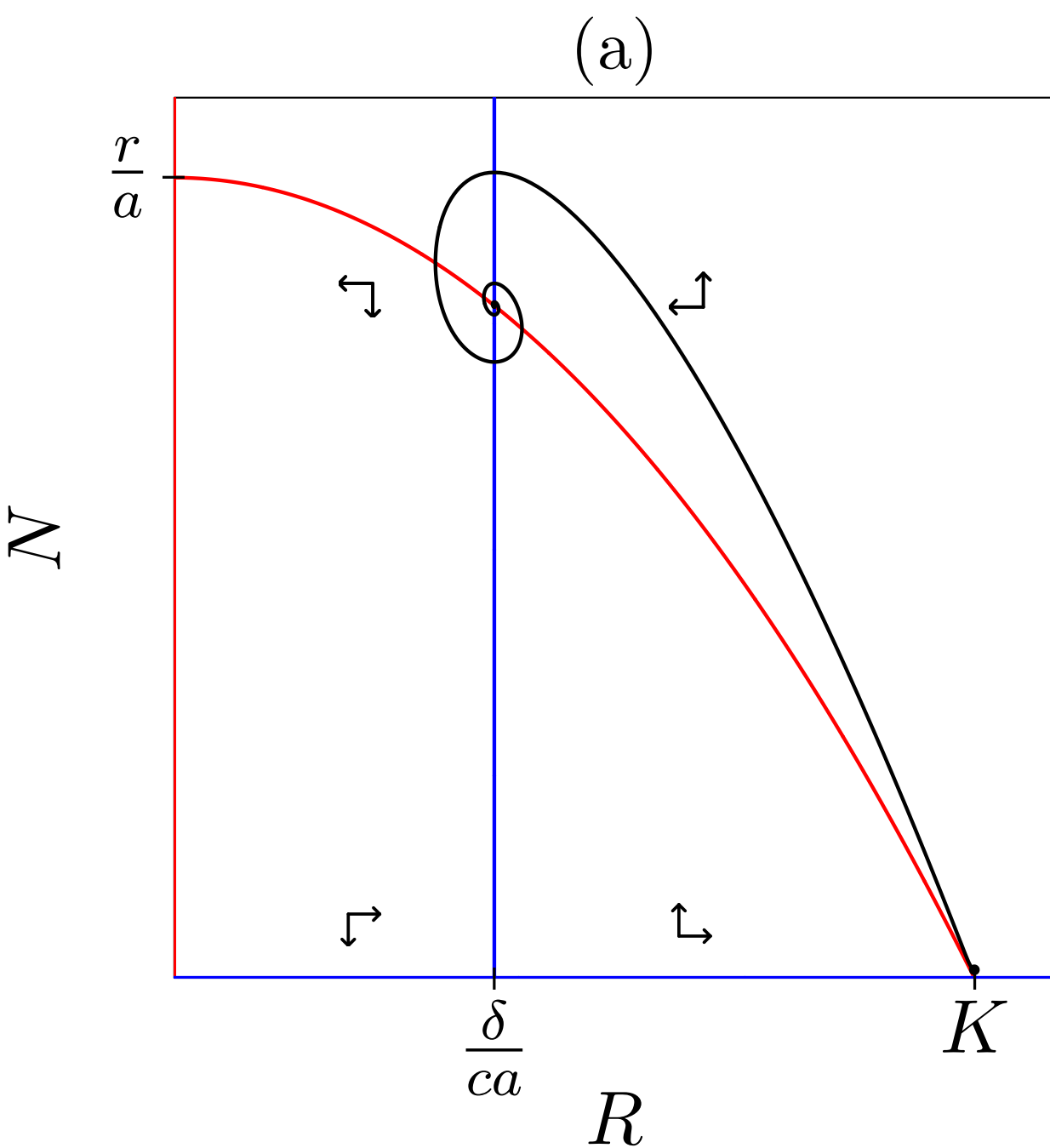
Generalized Lotka-Volterra model $\frac{dR}{dt} = rR(1 - (R/K)^m) - aRN$, $\frac{dN}{dt} = caRN - \delta N$



$$J = \begin{pmatrix} \partial_R f & \partial_N f \\ \partial_R g & \partial_N g \end{pmatrix} \Big|_{(\bar{R}, \bar{N})} = \begin{pmatrix} -\alpha & -\beta \\ +\gamma & 0 \end{pmatrix}$$

When to expect Lotka-Volterra like nullclines?

$$\frac{dR}{dt} = rR(1 - (R/K)^m) - aRN, \quad \frac{dN}{dt} = caRN - \delta N$$



$$\frac{dR}{dt} = 0 \quad \text{gives} \quad N = \frac{r}{a}(1 - (R/K)^m)$$

$$\frac{dR}{dt} = [f(R) - aN] R$$

$$\frac{dR}{dt} = 0 \quad \text{gives} \quad N = f(R)/c$$

$$\frac{dN}{dt} = \left[\frac{\beta R}{h + R} - \delta \right] N = [\beta f(R) - \delta] N$$

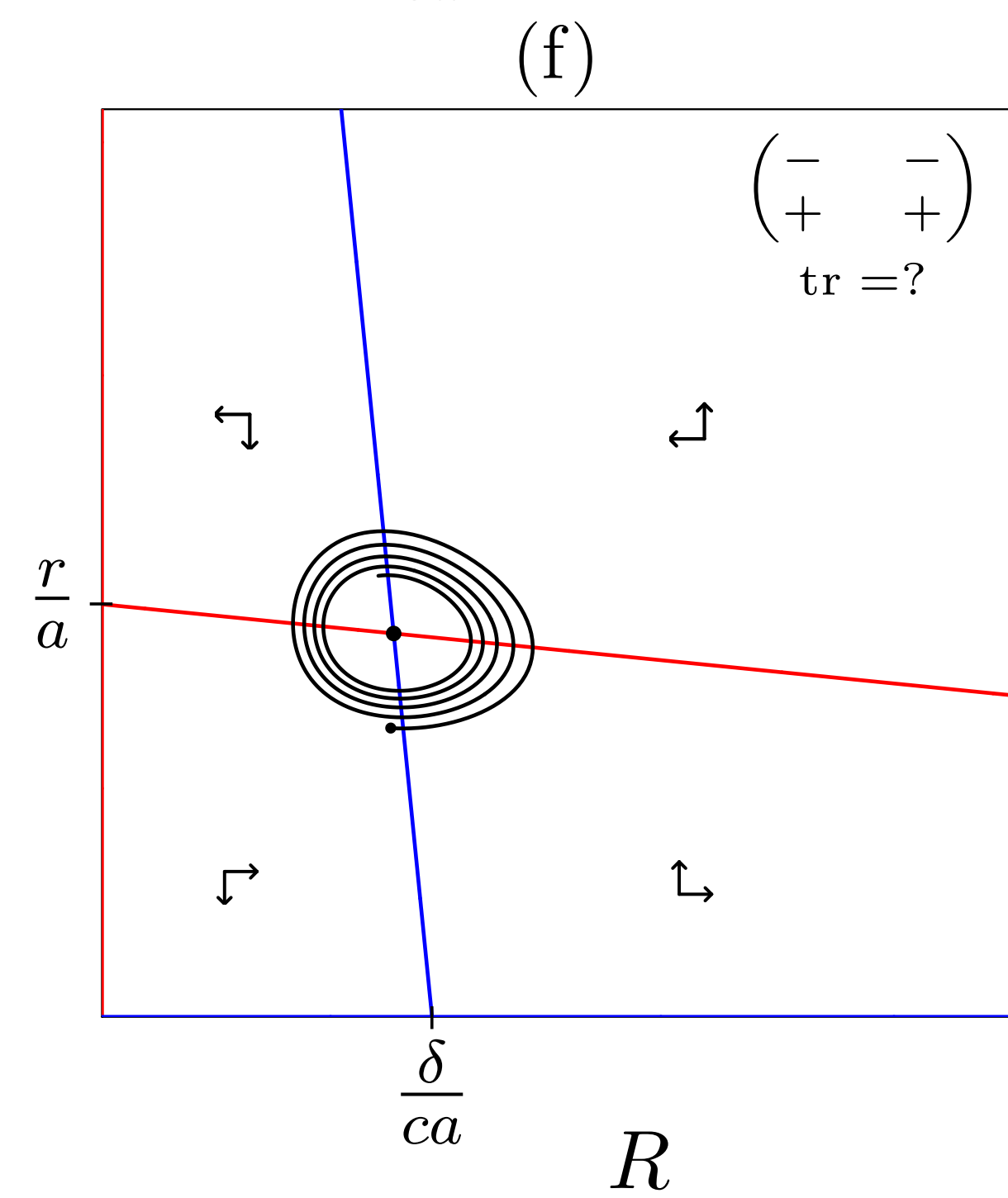
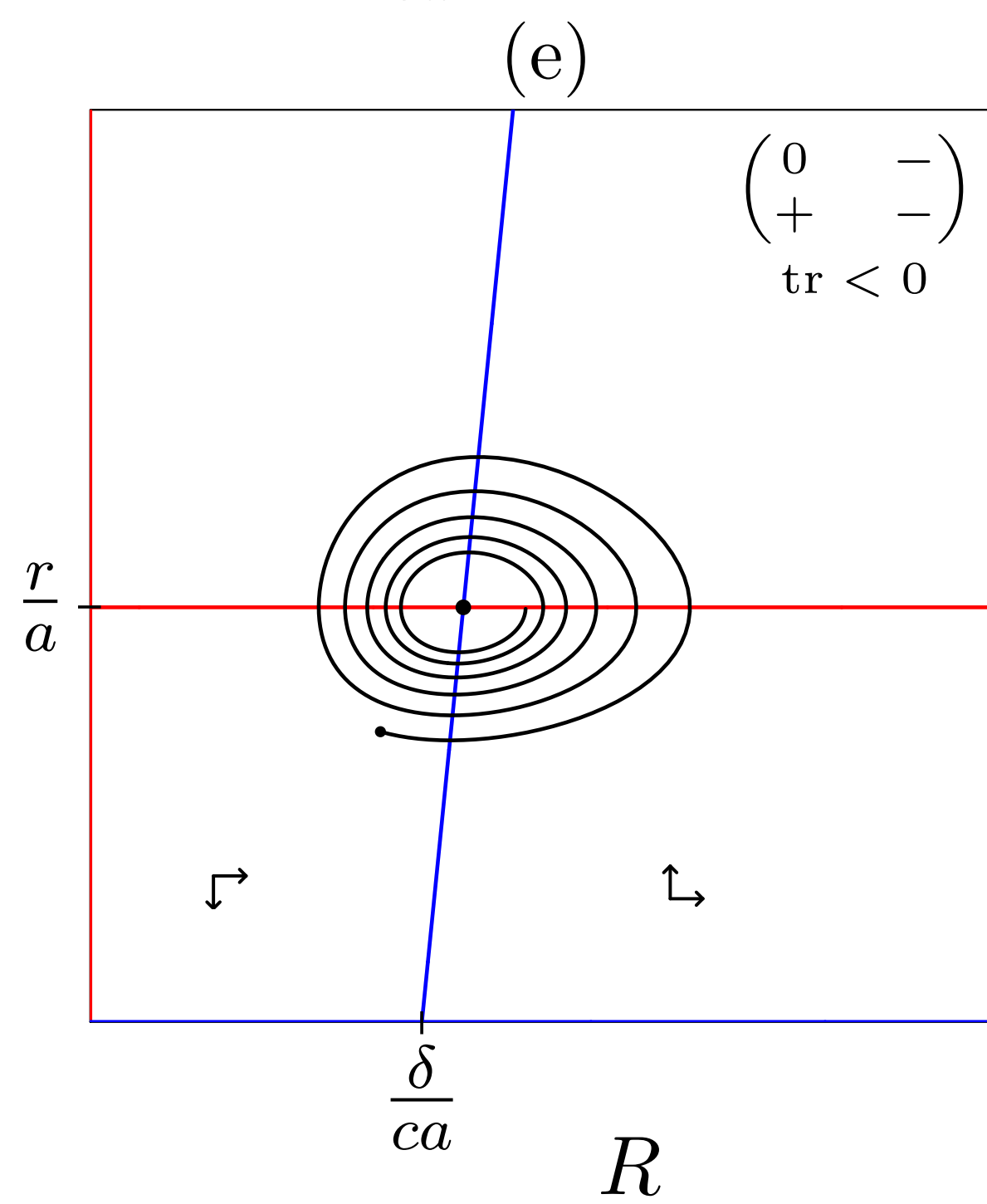
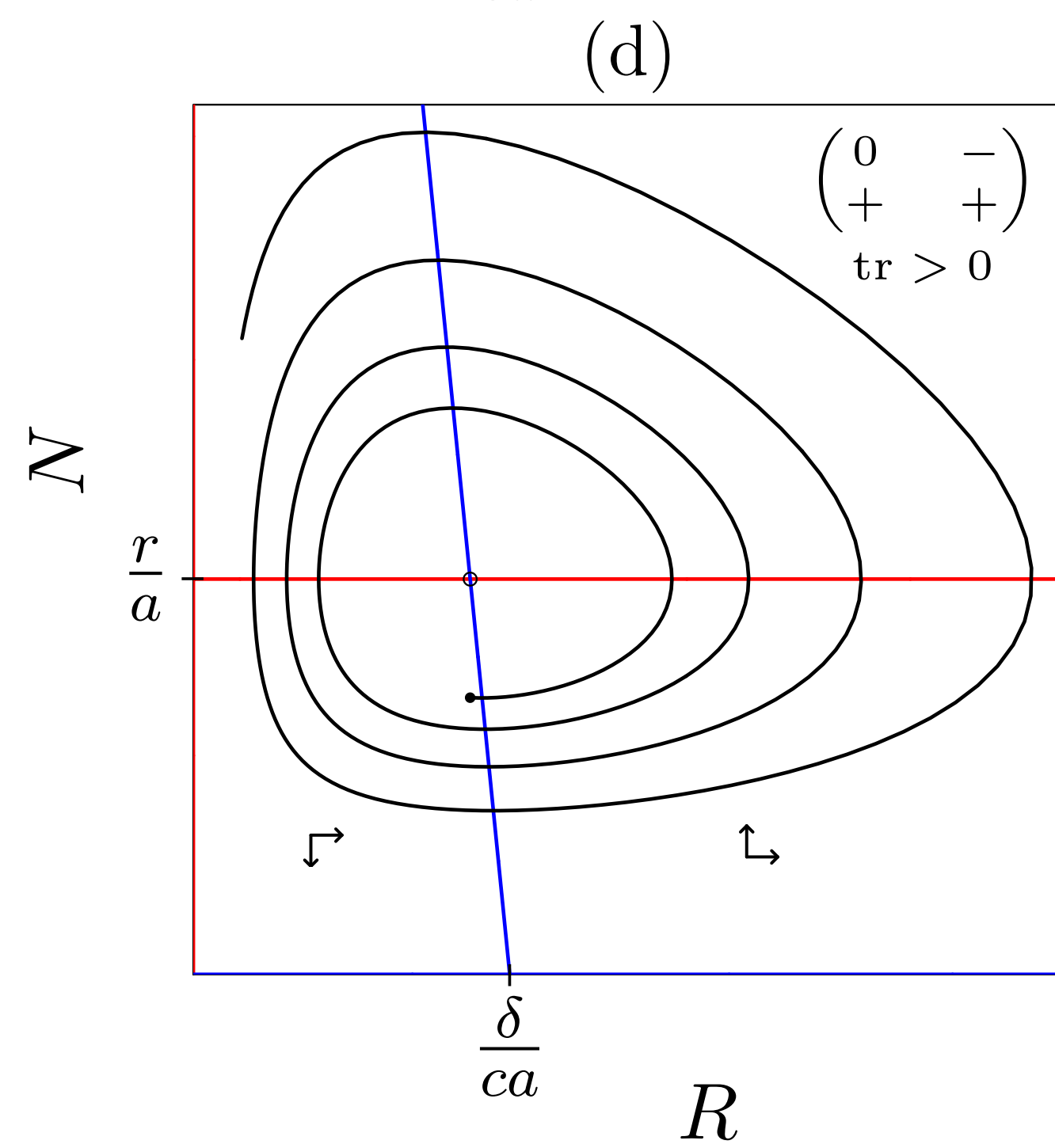
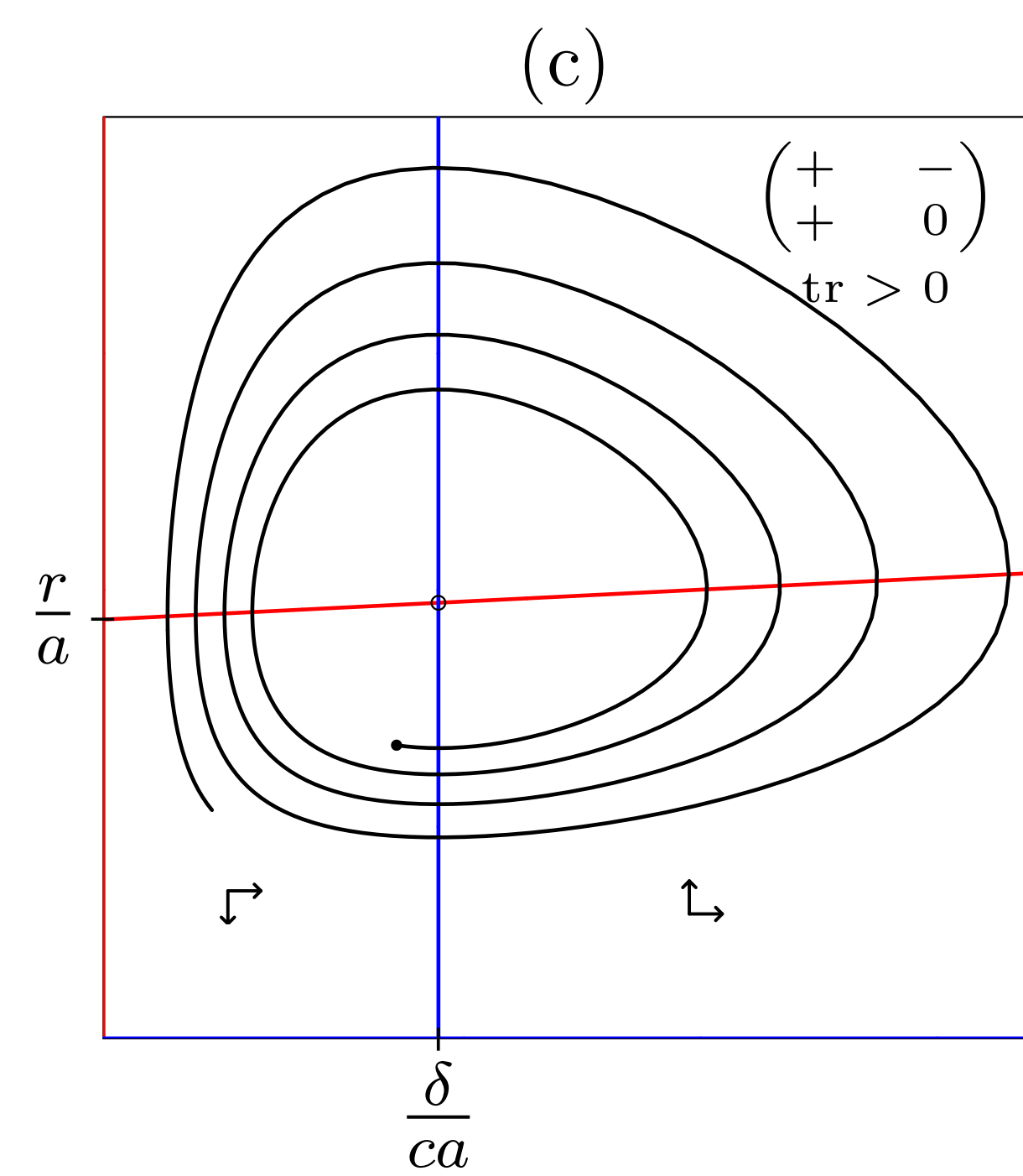
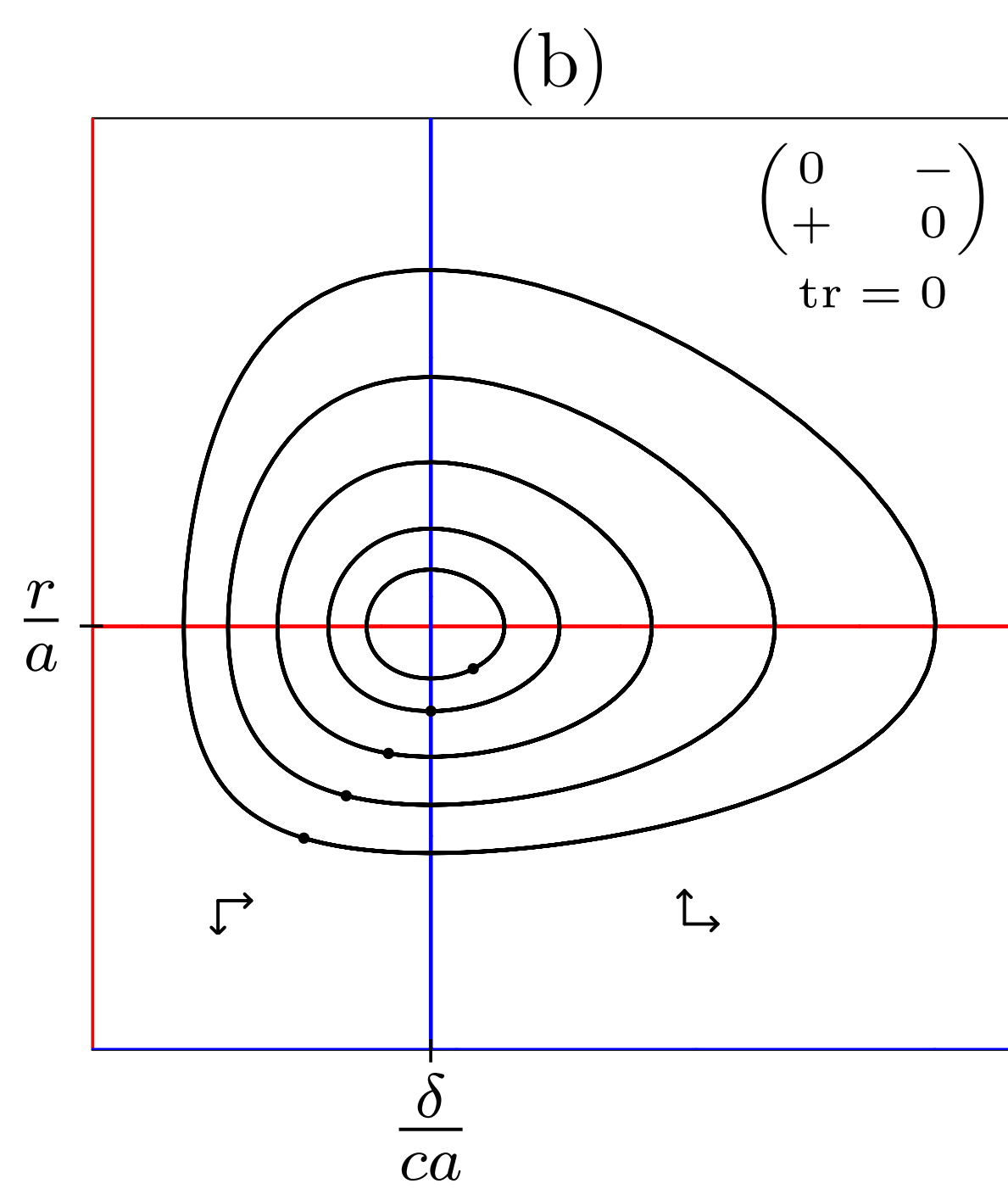
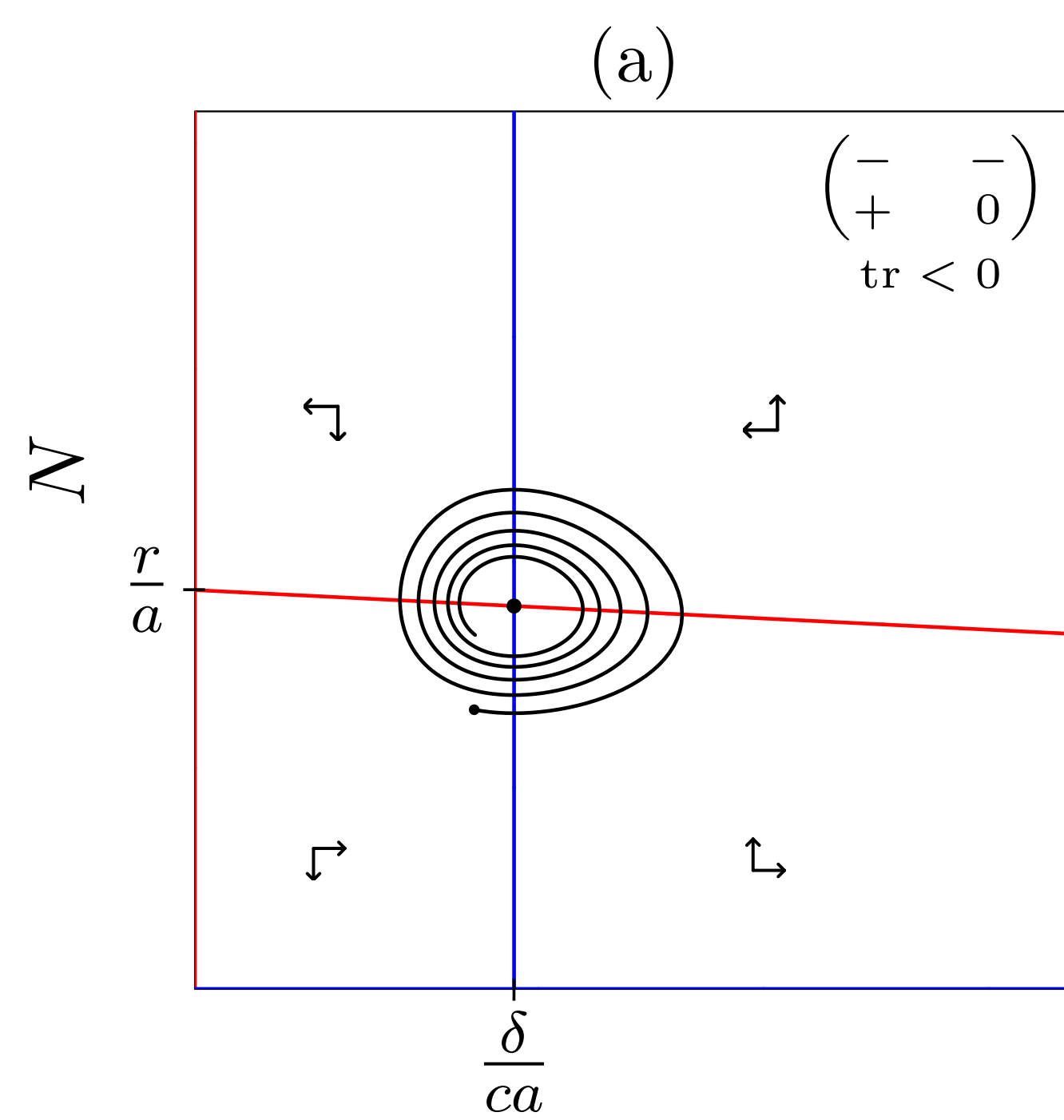
When are horizontal and vertical nullclines a robust result?

$$\frac{dR}{dt} = rR - aRN \quad \text{and} \quad \frac{dN}{dt} = caRN - \delta N$$

$$N = \frac{r}{a} \quad \text{and} \quad R = \frac{\delta}{ca}$$

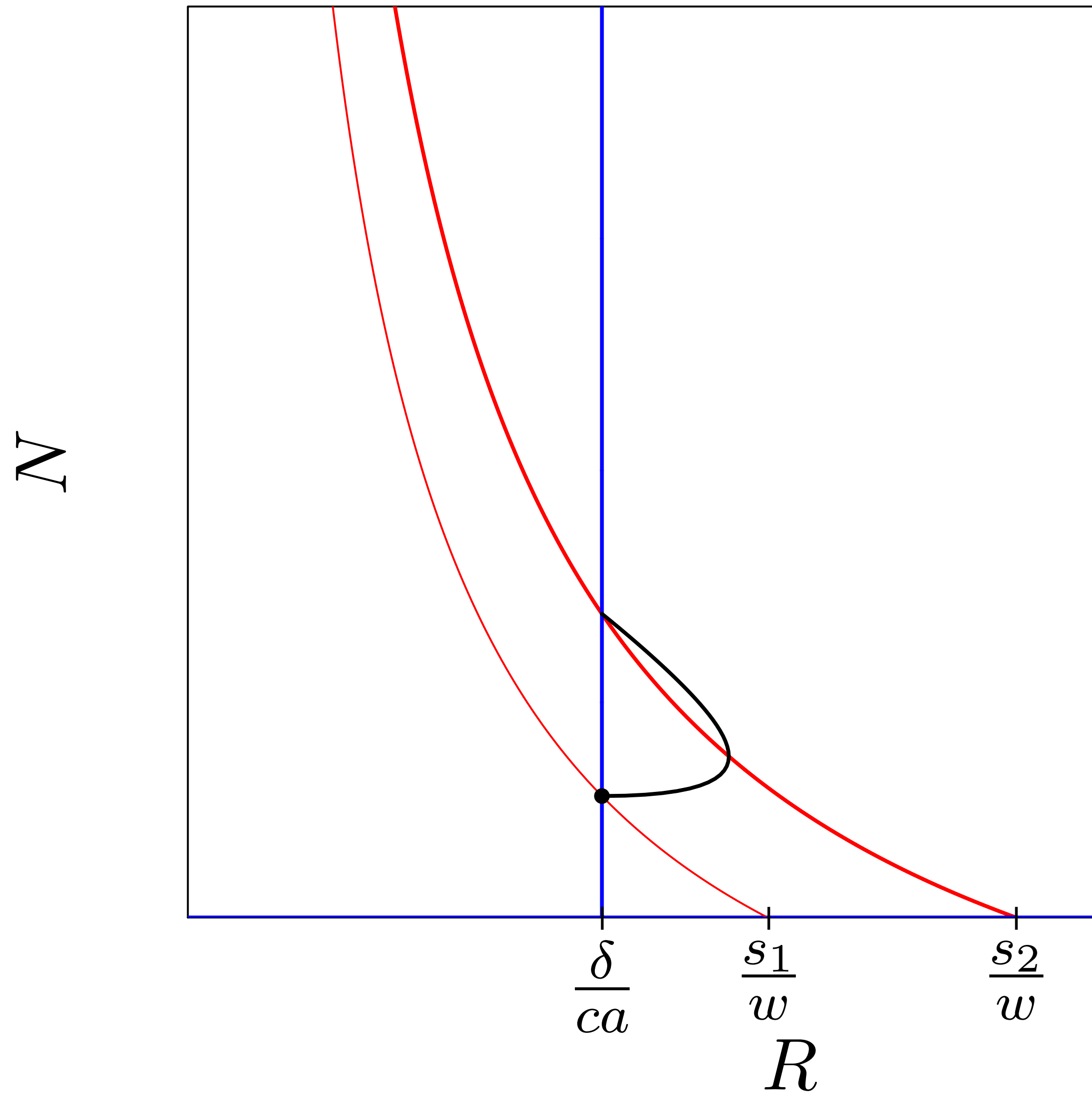
$$J = \begin{pmatrix} r - a\bar{N} & -a\bar{R} \\ ca\bar{N} & ca\bar{R} - \delta \end{pmatrix} = \begin{pmatrix} 0 & -\delta/c \\ cr & 0 \end{pmatrix}$$

$$\lambda_{\pm} = \pm \sqrt{-\delta r} = \pm i \sqrt{\delta r}$$



Enrichment for resource affects consumer steady state only

(a)



(b)

