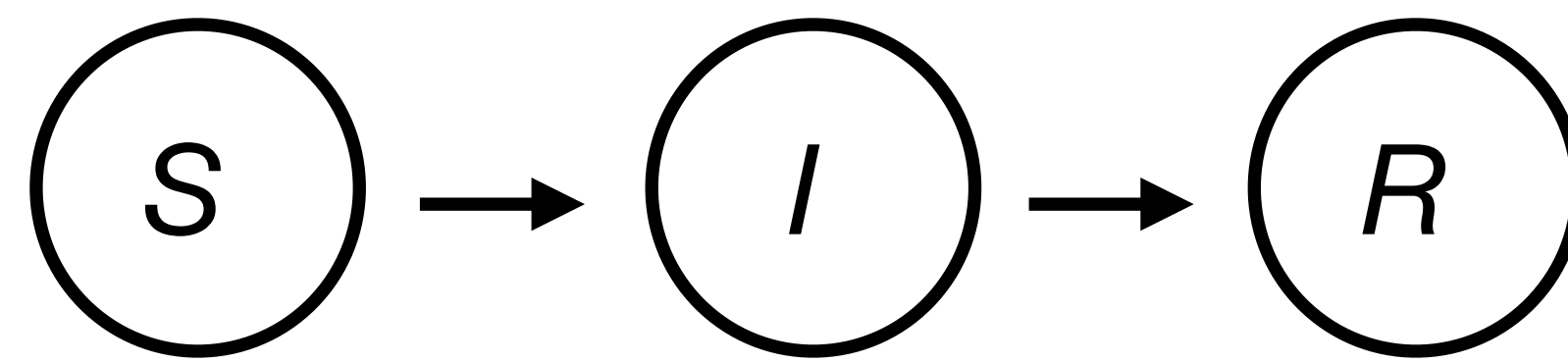


Chapter 6: R_0



SIR model:

$$\frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dI}{dt} = \beta SI - (\delta + r)I, \quad \text{and} \quad \frac{dR}{dt} = rI - dR,$$

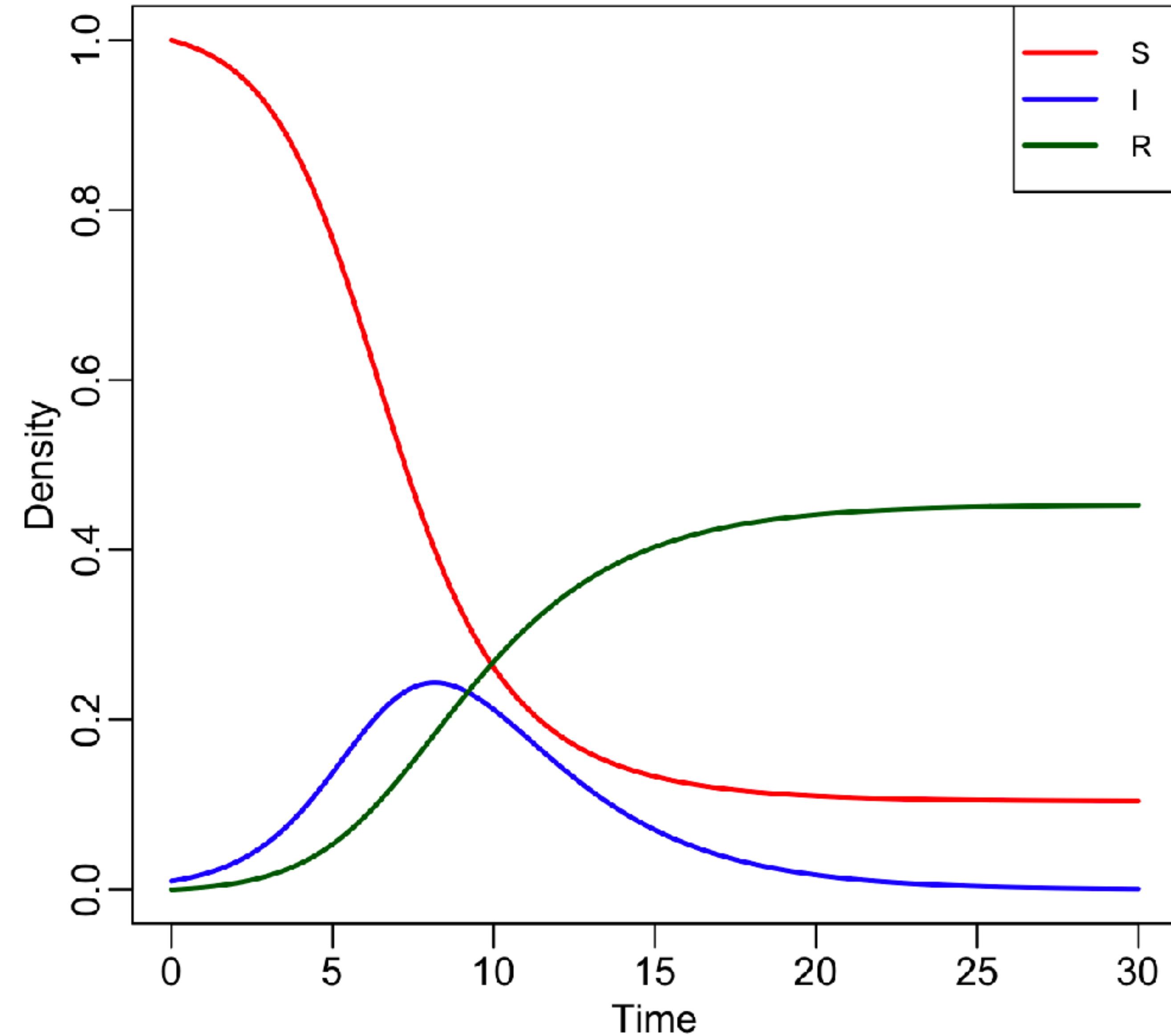
Virulence: $v = \delta - d$

$$\bar{S} = \frac{\delta + r}{\beta}, \quad \bar{I} = \frac{s}{\delta + r} - \frac{d}{\beta}, \quad \text{and} \quad \bar{R} = \frac{r}{d} \bar{I} = \frac{rs}{d(\delta + r)} - \frac{r}{\beta},$$

$$R_0 = \beta \bar{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}$$

An epidemic dies out when the susceptibles drop below a threshold

$$\frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dI}{dt} = \beta SI - (\delta + r)I, \quad \text{and} \quad \frac{dR}{dt} = rI - dR, \quad R_0 = \beta \bar{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}$$



$$\beta SI = (\delta + r)I$$

$$S = \frac{\delta + r}{\beta} = \frac{\bar{S}}{R_0}$$

Herd immunity

$$S(0) = 1, \quad s = d = 0, \quad r = \delta = 0.2, \quad \beta = 1$$

R_0 can also be computed from Jacobian

SIR model:

$$\frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dI}{dt} = \beta SI - (\delta + r)I, \quad \text{and} \quad \frac{dR}{dt} = rI - dR,$$

$$R_0 = \beta \bar{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}$$

$$\text{SI model } (I=0): J = \begin{pmatrix} -d & -\beta \bar{S} \\ 0 & \beta \bar{S} - \delta - r \end{pmatrix}$$

$$\lambda_1 > 0 \text{ same as } R_0 > 1$$

$$\lambda_1 = \beta \bar{S} - \delta - r \text{ and } \lambda_2 = -d.$$

R_0 is not equal to the growth/expansion rate of the epidemic

SIR model:

$$\frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dI}{dt} = \beta SI - (\delta + r)I, \quad \text{and} \quad \frac{dR}{dt} = rI - dR, \quad R_0 = \beta \bar{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}$$

Initial expansion rate:

$$\frac{dI}{dt} = \beta \bar{S} I - (\delta + r)I = \left(\frac{\beta s}{d} - \delta - r \right) I = \rho_0 I$$

$$\lambda_1 = \beta \bar{S} - \delta - r$$

R_0 is not equal to the growth/expansion rate of the epidemic

$$\frac{dI}{dt} = \beta \bar{S} I - (\delta + r) I = \left(\frac{\beta s}{d} - \delta - r \right) I = \rho_0 I \quad \text{vs.} \quad R_0 = \beta \bar{S} \frac{1}{\delta + r} = \frac{s}{d} \frac{\beta}{\delta + r}$$

Define length of the infectious period, $L = \frac{1}{\delta + r}$ to see that

$$R_0 = \beta \bar{S} L \quad \text{and} \quad \rho_0 = \beta \bar{S} - \frac{1}{L} \quad \text{such that} \quad \beta \bar{S} = \rho_0 + \frac{1}{L} \quad \text{and} \quad R_0 = \rho_0 L + 1$$

From the latter:

$$\rho_0 = \frac{R_0 - 1}{L} = (R_0 - 1)(\delta + r)$$

Frequency dependent infections

SIR model:

$$\frac{dS}{dt} = s - dS - \beta SI , \quad \frac{dI}{dt} = \beta SI - (\delta + r)I , \quad \text{and} \quad \frac{dR}{dt} = rI - dR ,$$

$$R_0 = \frac{s}{d} \frac{\beta}{\delta + r}$$

Frequency dependent infection in SIR model: $N = S + I + R$

$$\frac{dS}{dt} = s - dS - \frac{\beta SI}{N} , \quad \frac{dI}{dt} = \frac{\beta SI}{N} - (\delta + r)I , \quad \text{and} \quad \frac{dR}{dt} = rI - dR ,$$

$$R_0 = \frac{\beta}{\delta + r}$$

SEIR model

$$\frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dE}{dt} = \beta SI - (\gamma + d)E, \quad \frac{dI}{dt} = \gamma E - (\delta + r)I, \quad \frac{dR}{dt} = rI - dR,$$

↓ 3
↓ 2
↓ 1

$$\bar{I} = \frac{s}{\beta \bar{S}} - \frac{d}{\beta} = \frac{\gamma}{\gamma + d} \frac{s}{\delta + r} - \frac{d}{\beta} \quad \leftarrow \quad \bar{S} = \frac{\gamma + d}{\gamma} \frac{\delta + r}{\beta} \quad \leftarrow \quad \bar{E} = \frac{\delta + r}{\gamma} I$$

$$R_0 = \frac{s}{d} \frac{\beta}{\delta + r} \frac{\gamma}{\gamma + d}.$$

$R_0 > 1$ is the same as $\bar{I} > 0$

SEIR model

$$R_0 = \frac{s}{d} \frac{\beta}{\delta + r} \frac{\gamma}{\gamma + d} .$$

$$\frac{dS}{dt} = s - dS - \beta SI \ , \ \frac{dE}{dt} = \beta SI - (\gamma + d)E \ , \ \frac{dI}{dt} = \gamma E - (\delta + r)I \ , \ \frac{dR}{dt} = rI - dR \ ,$$

$$J = \begin{pmatrix} \partial_E E' & \partial_I E' \\ \partial_E I' & \partial_I I' \end{pmatrix} = \begin{pmatrix} -(\gamma + d) & \beta \bar{S} \\ \gamma & -(\delta + r) \end{pmatrix}$$

$$\text{tr} < 0 \text{ and } \det = (\gamma + d)(\delta + r) - \gamma\beta\bar{S}$$

$$\det J < 0 \quad \text{or} \quad \frac{d}{s} \frac{\gamma + d}{\gamma} \frac{\delta + r}{\beta} - 1 < 0 \quad \Leftrightarrow \quad \frac{1}{R_0} < 1 \quad \Leftrightarrow \quad R_0 > 1$$

Fitness in consumer-resource models

$$\frac{dR}{dt} = bR(1 - R/k) - dR - aRN \quad \text{and} \quad \frac{dN}{dt} = caRN - \delta N$$

$$K = k(1 - d/b) \quad R_{0_R} = b/d. \quad R_{0_N} = caK/\delta$$

$$\frac{dR}{dt} = bR(1 - R/k) - dR - aRN \quad \text{and} \quad \frac{dN}{dt} = \frac{\beta RN}{h + R} - \delta N$$

$$R_0 = \frac{\beta K}{\delta(h+K)} \quad \text{or} \quad R_0 = \beta/\delta$$

$$K = k(1 - d/b)$$