

Chapter 8: Modeling chains of ODEs

$$\frac{dR}{dt} = [r(1 - R/K) - bN]R, \quad \frac{dN}{dt} = [bR - d - cM]N \quad \text{and} \quad \frac{dM}{dt} = [cN - e]M$$

$n=1$ $\bar{R} = K$

$n=2$ $\bar{R} = \frac{d}{b}$ and $\bar{N} = \frac{r}{b} \left(1 - \frac{d}{bK} \right) = \frac{r}{b} \left(1 - \frac{1}{R_0} \right)$

Modeling chains of ODEs

$$\frac{dR}{dt} = [r(1 - R/K) - bN]R, \quad \frac{dN}{dt} = [bR - d - cM]N \quad \text{and} \quad \frac{dM}{dt} = [cN - e]M$$

$$n=1 \quad \bar{R} = K$$

$$n=2 \quad \bar{R} = \frac{d}{b} \quad \text{and} \quad \bar{N} = \frac{r}{b} \left(1 - \frac{d}{bK} \right) = \frac{r}{b} \left(1 - \frac{1}{R_0} \right)$$

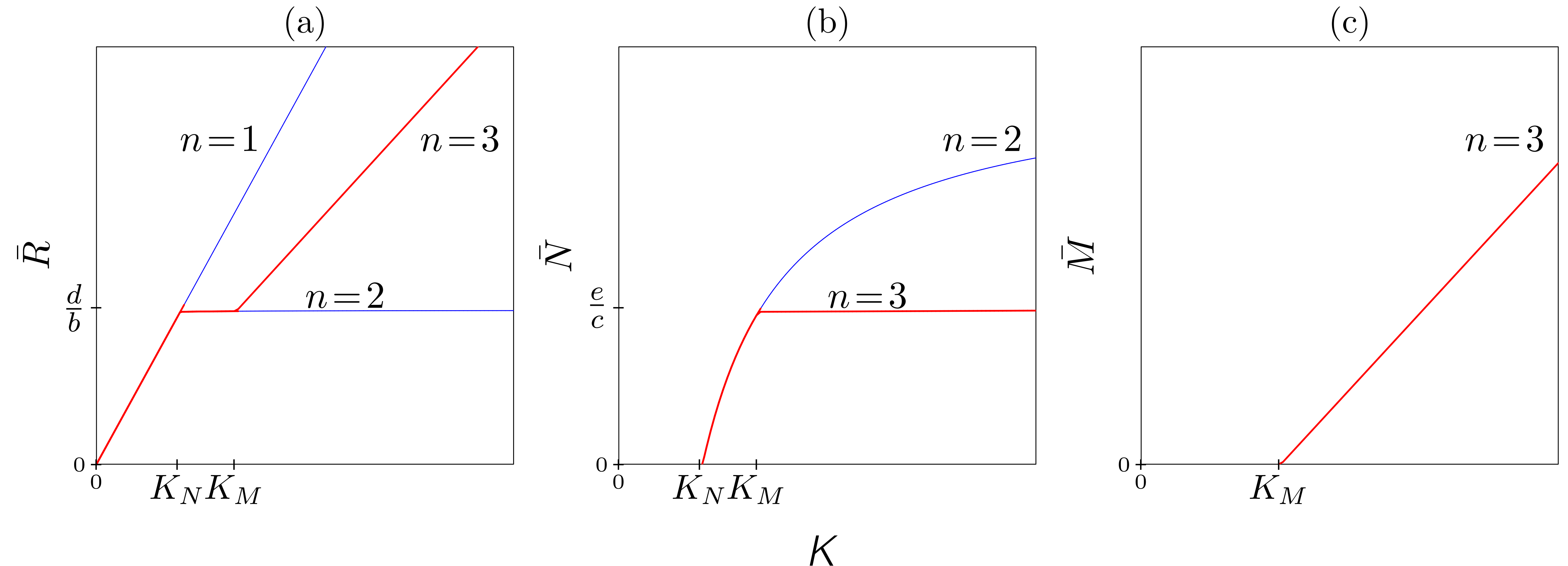
$$R'_0 = \frac{cr}{be}$$

$$n=3 \quad \bar{N} = \frac{e}{c}, \quad \bar{R} = K \left(1 - \frac{be}{cr} \right) \quad \text{and} \quad \bar{M} = \frac{b\bar{R} - d}{c}$$

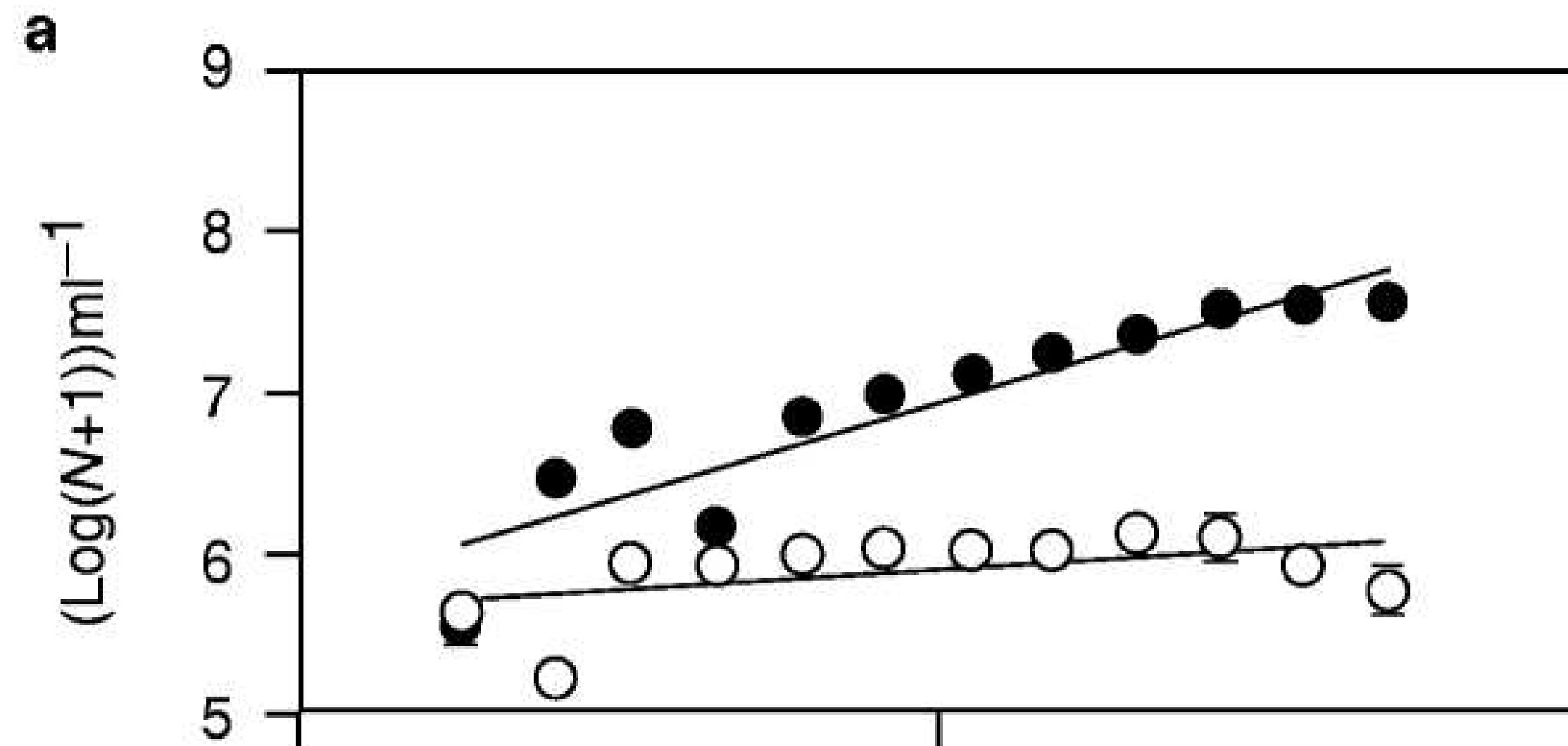
For odd chain lengths R depends on K

Modeling chains of ODEs

$$\frac{dR}{dt} = [r(1 - R/K) - bN]R, \quad \frac{dN}{dt} = [bR - d - cM]N \quad \text{and} \quad \frac{dM}{dt} = [cN - e]M$$

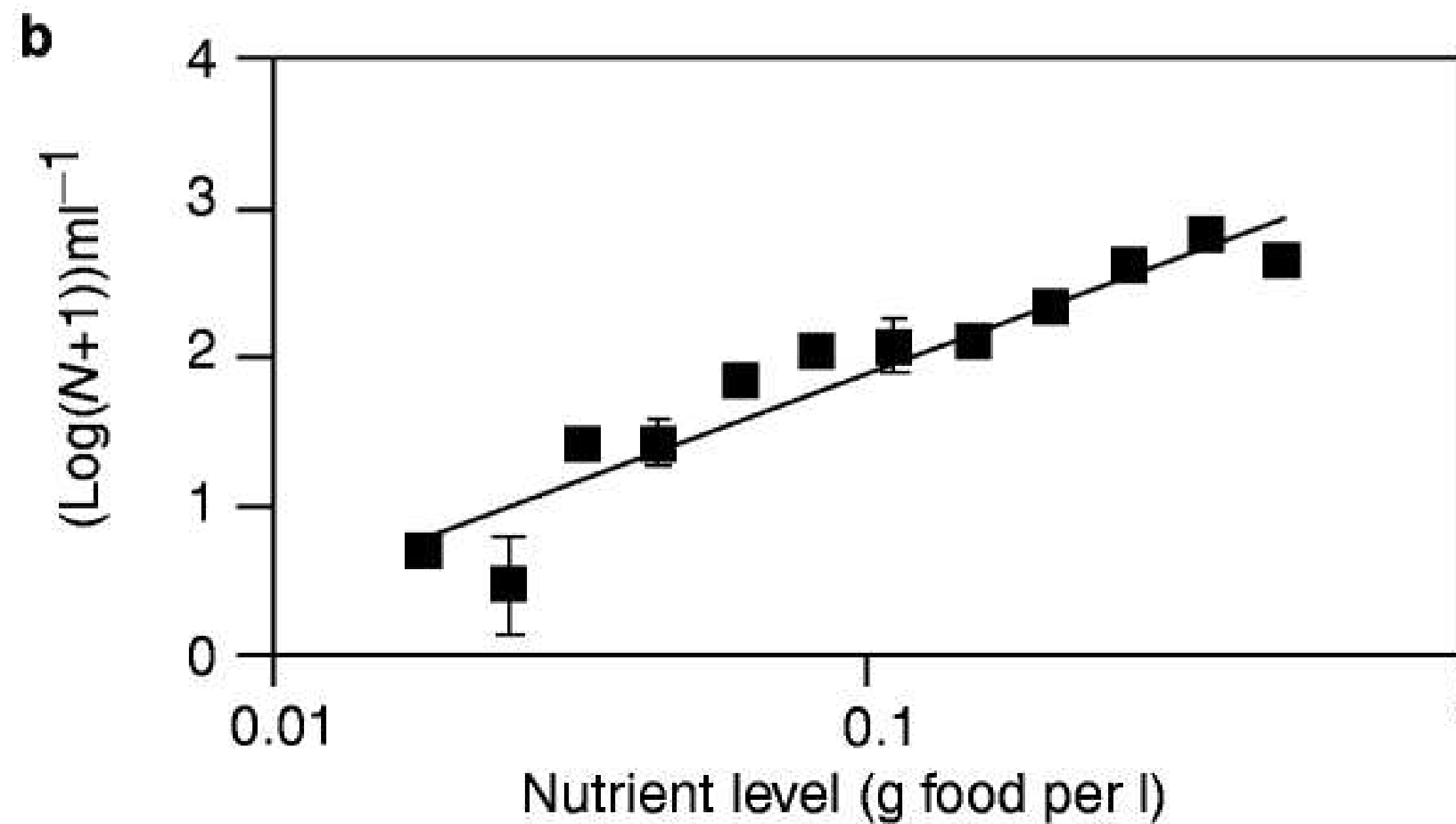


Kaunzinger & Morin, Nature, 1998



Predator

Prey with predator



Prey alone

Bacterial food chain

Modeling chains with saturated interaction terms

$$\frac{dR}{dt} = \left[r \left(1 - \frac{R}{K} \right) - \frac{bN}{h_R + R} \right] R, \quad \frac{dN}{dt} = \left[\frac{bR}{h_R + R} - d - \frac{cM}{h_N + N} \right] N, \text{ and } \frac{dM}{dt} = \left[\frac{cN}{h_N + N} - e \right] M$$

f_R: R and N

f_N: in absence of M no N

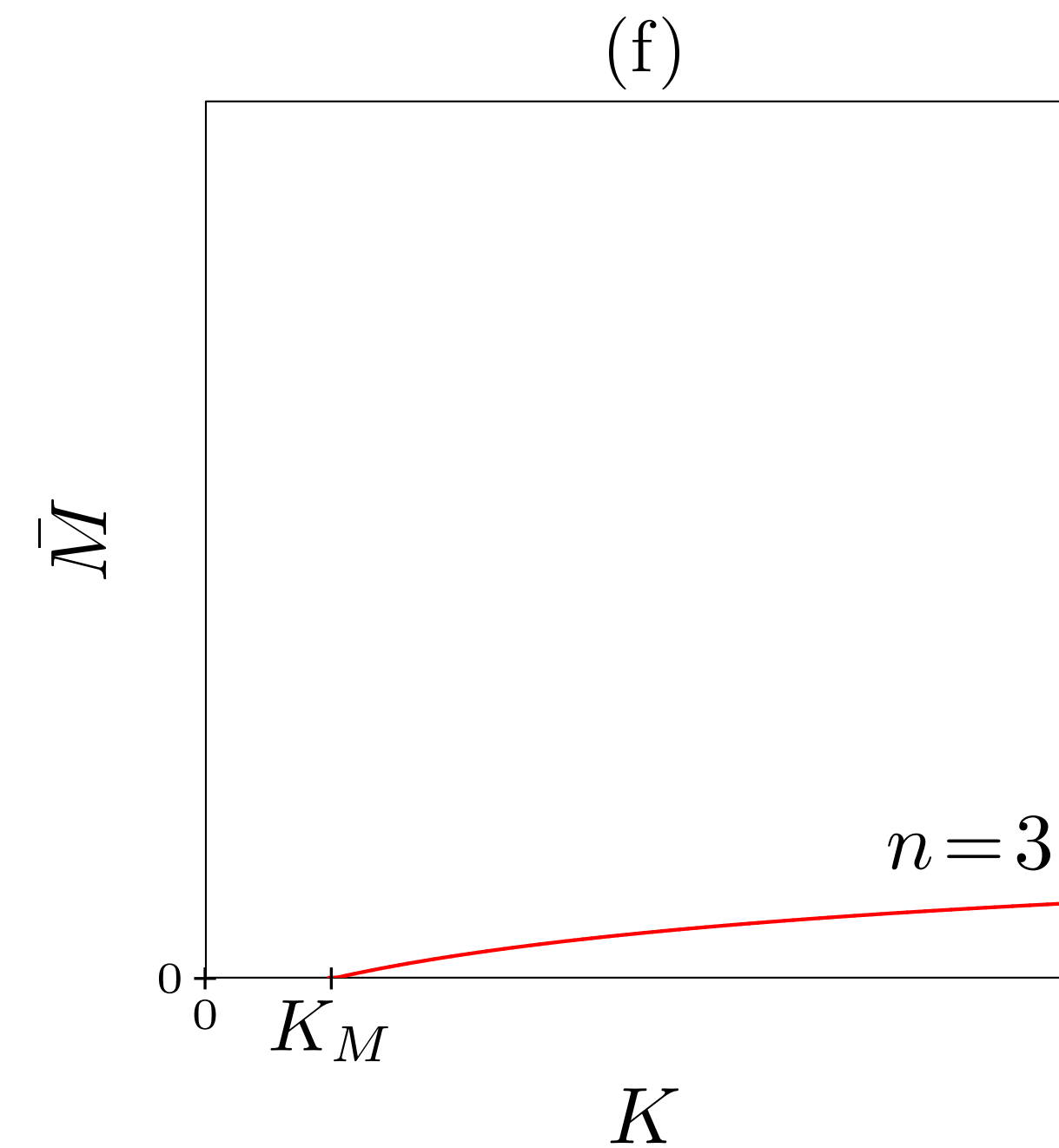
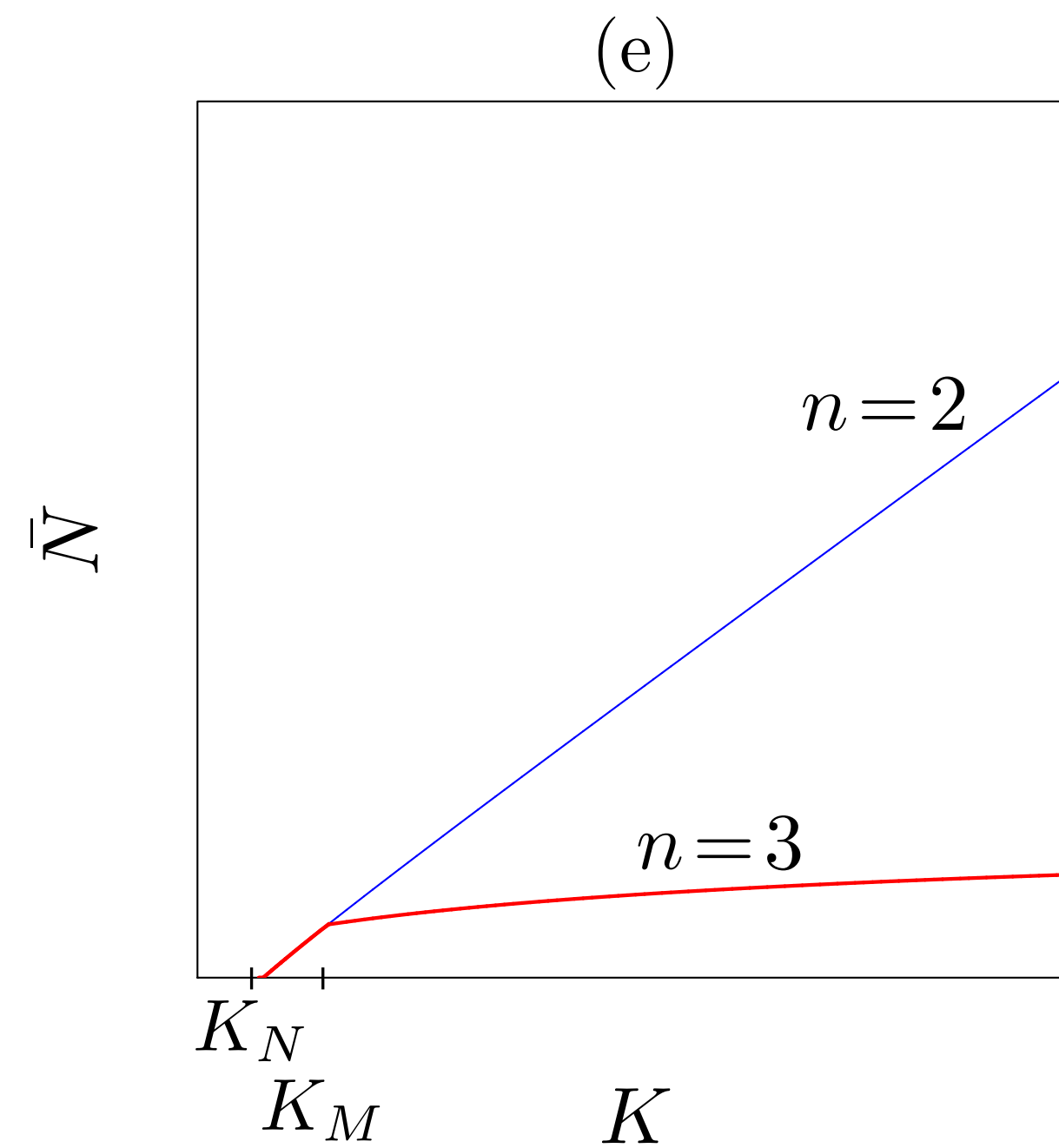
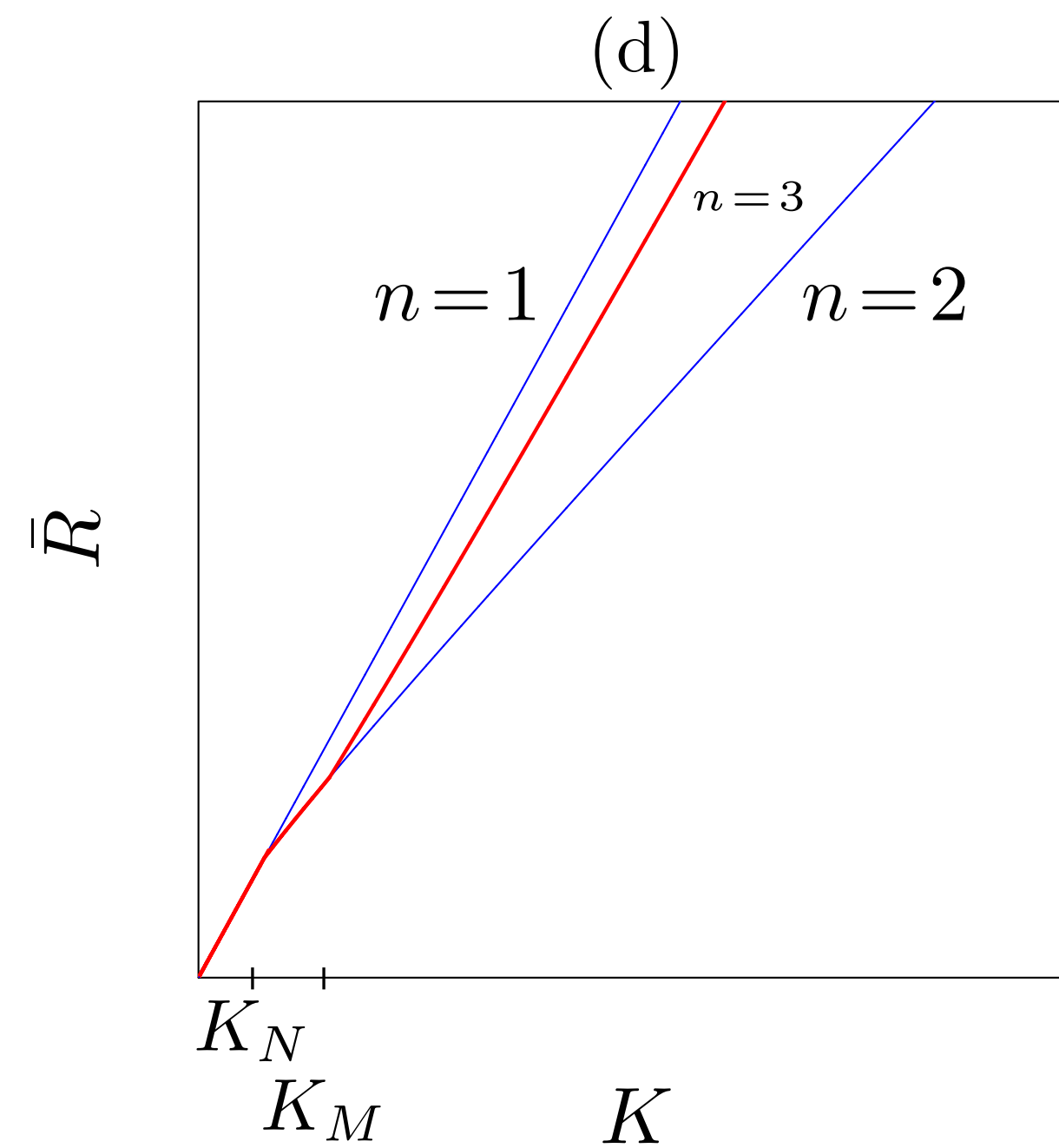
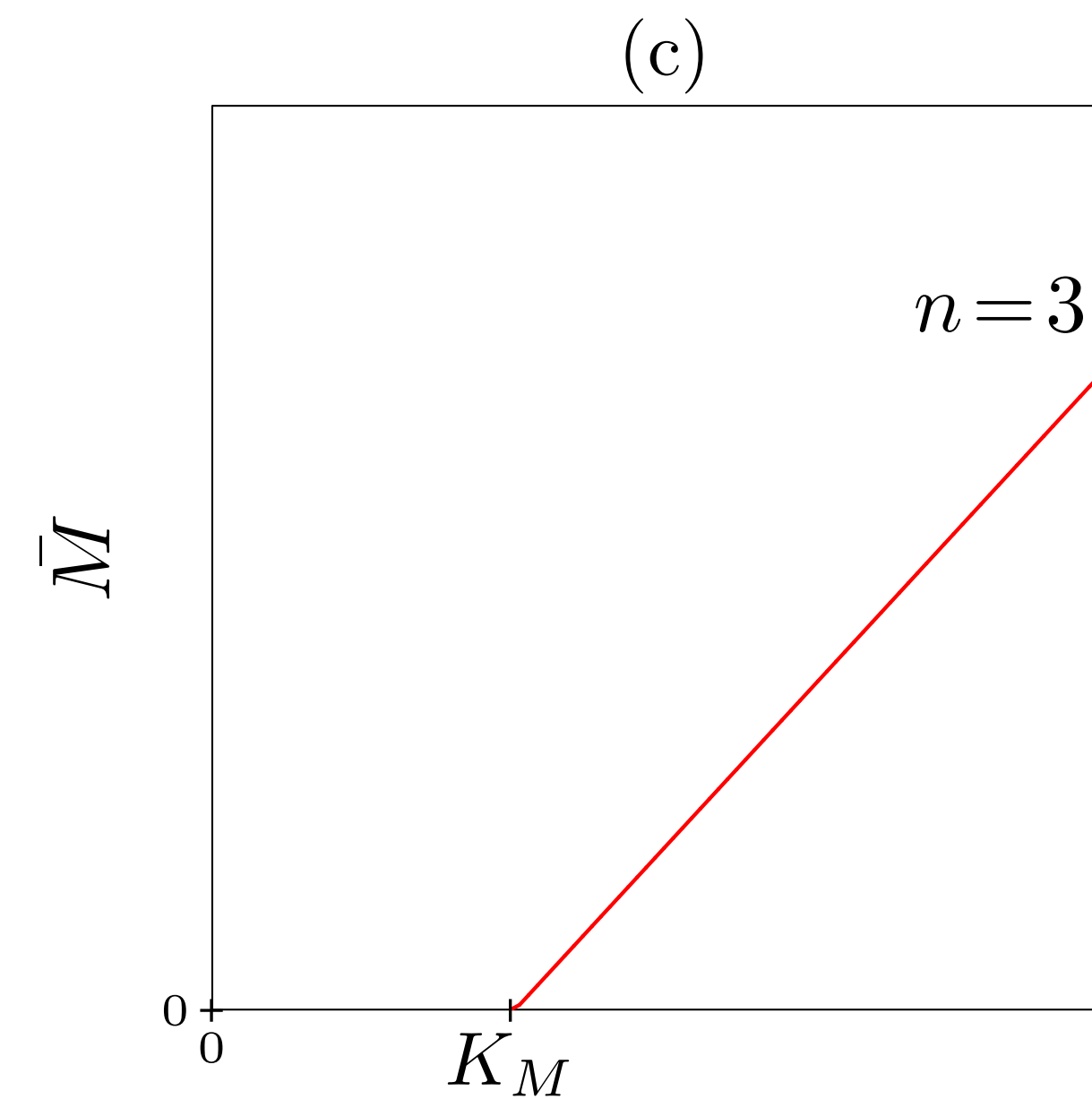
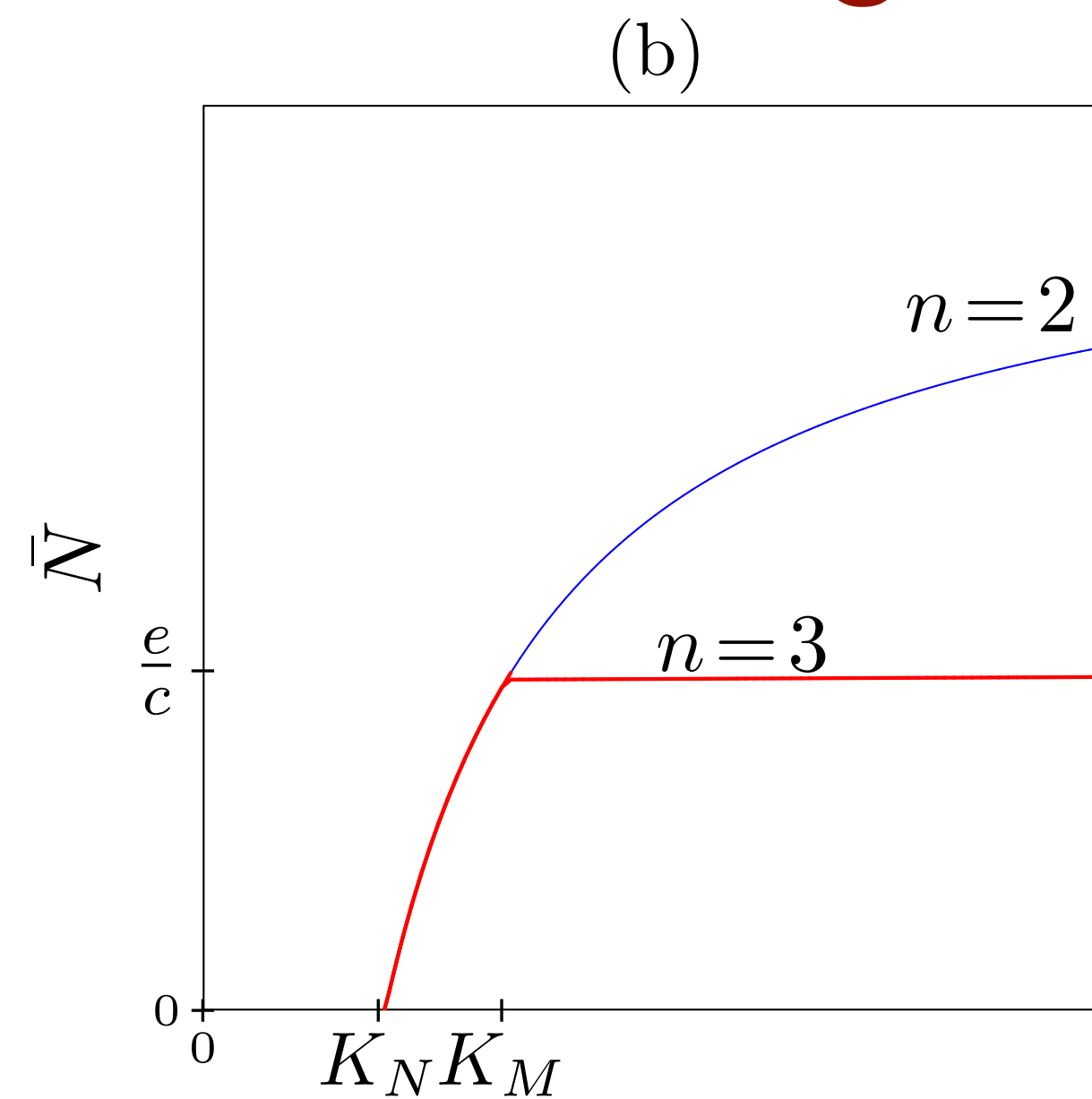
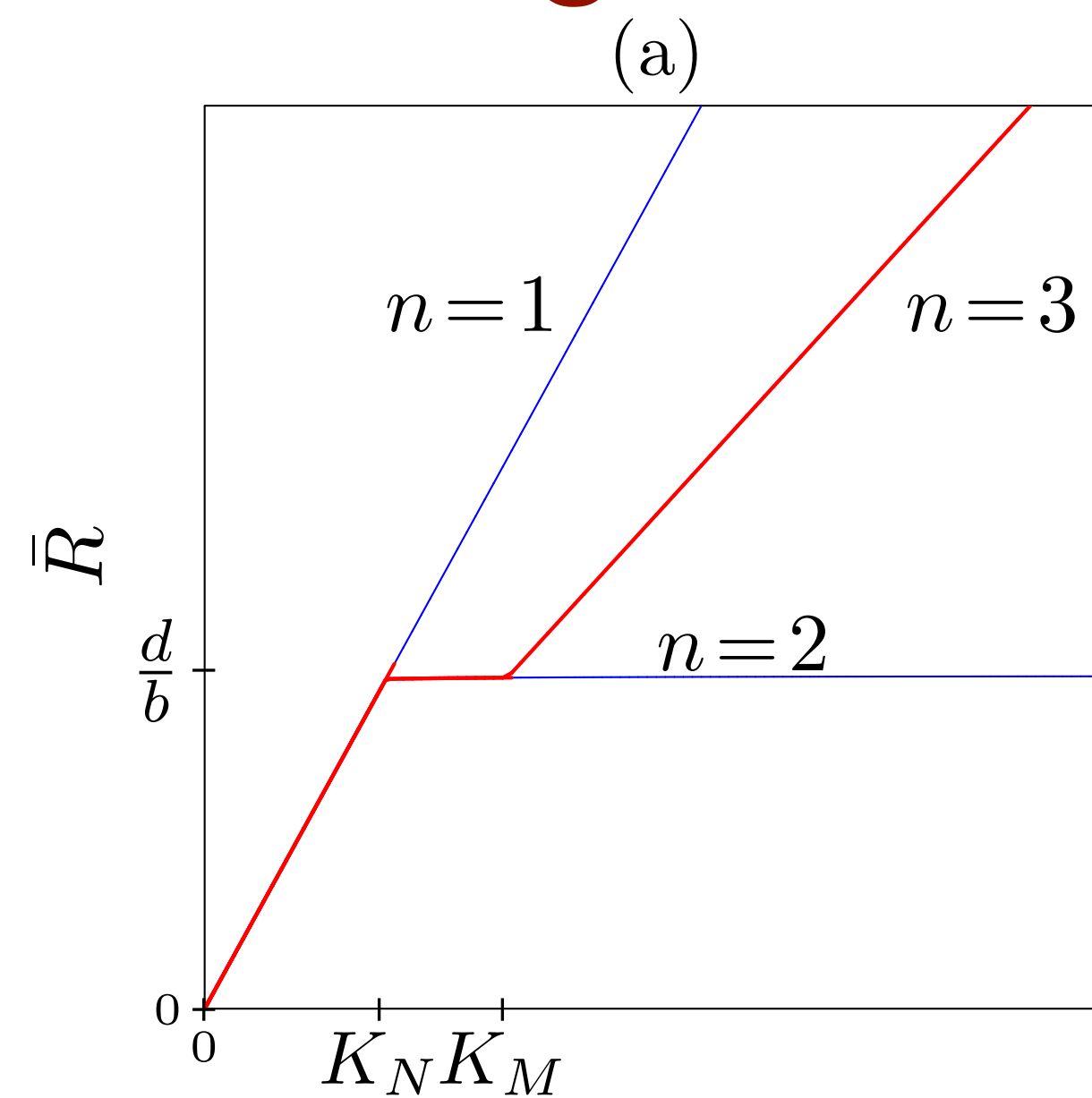
f_M: no M

$$\frac{dR}{dt} = \left[r \left(1 - \frac{R}{K} \right) - \frac{bN}{h_R + R + N} \right] R, \quad \frac{dN}{dt} = \left[\frac{bR}{h_R + R + N} - d - \frac{cM}{h_N + N + M} \right] N \quad \text{and} \quad \frac{dM}{dt} = \left[\frac{cN}{h_N + N + M} - e \right] M$$

Per capita function always depends on variable itself.

$$aXY \simeq \frac{aXY}{1 + X/k + Y/k} \quad \text{when } k \text{ is large}$$

Modeling chains with Beddington interaction terms



Other famous chains don't suffer from this problem

SEIR model:

$$\frac{dS}{dt} = s - dS - \beta SI, \quad \frac{dE}{dt} = \beta SI - (d + \gamma)E, \quad \frac{dI}{dt} = \gamma E - (\delta + r)I \quad \text{and} \quad \frac{dR}{dt} = rI - dR$$

$$\bar{R} = \frac{r}{d} \bar{I}, \quad \bar{I} = \frac{\gamma}{\delta + r} \bar{E}, \quad \bar{S} = \frac{(d + \gamma)(\delta + r)}{\gamma\beta}, \quad \bar{E} = \frac{s}{d + \gamma} - \frac{d(\delta + r)}{\gamma\beta}$$

$\bar{R}$ and $\bar{I}$ are proportional to previous level

Cascade of cell divisions

$$\frac{dN_0}{dt} = s - (p + d)N_0 \quad , \quad \frac{dN_i}{dt} = 2pN_{i-1} - (p + d)N_i \quad \text{and} \quad \frac{dN_n}{dt} = 2pN_{n-1} - dN_n \quad ,$$

$$\bar{N}_0 = \frac{s}{p + d} \quad , \quad \bar{N}_i = \frac{2p}{p + d} \bar{N}_{i-1} \quad \text{and} \quad \bar{N}_n = \frac{2p}{d} \bar{N}_{n-1}$$

$$J = \begin{pmatrix} -(p + d) & 0 & 0 & 0 & \dots & \dots & 0 \\ 2p & -(p + d) & 0 & 0 & \dots & \dots & 0 \\ 0 & 2p & -(p + d) & 0 & \dots & \dots & 0 \\ & & \vdots & & & & \\ 0 & \dots & 0 & \dots & 0 & 2p & -d \end{pmatrix}$$

$$(J_{00} - \lambda)(J_{11} - \lambda)(J_{22} - \lambda) \dots (J_{nn} - \lambda) = 0$$

Solve eigenvalues from determinant (product diagonal elements)

Cascade of cell divisions

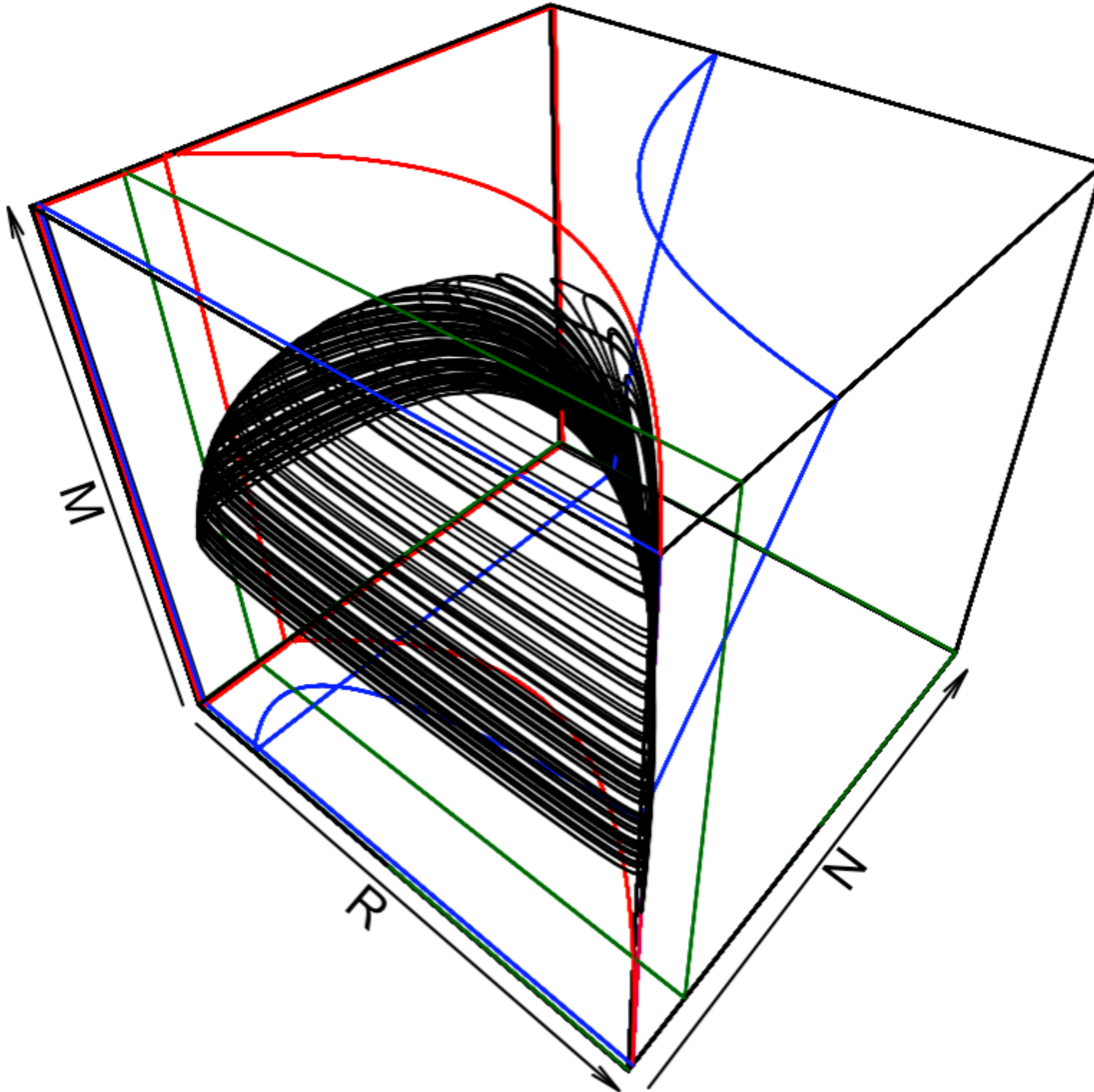
$$\frac{dN_0}{dt} = s - (p + d)N_0 \quad , \quad \frac{dN_i}{dt} = 2pN_{i-1} - (p + d)N_i \quad \text{and} \quad \frac{dN_n}{dt} = 2pN_{n-1} - dN_n \quad ,$$

$$\bar{N}_0 = \frac{s}{p + d} \quad , \quad \bar{N}_i = \frac{2p}{p + d} \bar{N}_{i-1} \quad \text{and} \quad \bar{N}_n = \frac{2p}{d} \bar{N}_{n-1}$$

$$\bar{N}_0 = \frac{s}{p + d} \quad , \quad \bar{N}_i = \frac{2^i p^i s}{(p + d)^{i+1}} \quad \text{and} \quad \bar{N}_n = \frac{s}{d} \left(\frac{2p}{p + d} \right)^n$$

$$\frac{dQ}{dt} = -aQ - d_Q Q + d \sum f_i N_i \quad \text{and} \quad s = aQ$$

Chaos in a 3D food chain



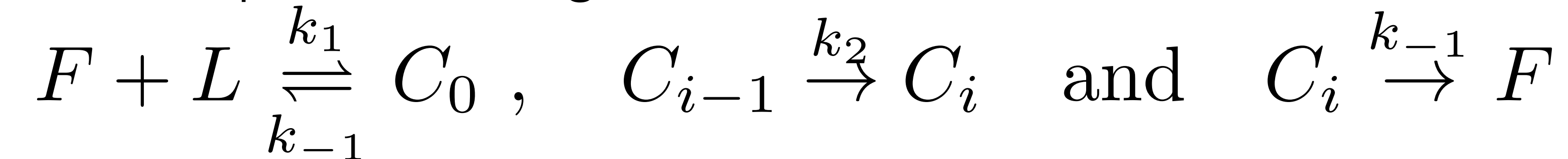
$$\begin{aligned}\frac{dR}{dt} &= R(1 - R) - c_1 N f(R) , \\ \frac{dN}{dt} &= -a_N N + c_1 N f(R) - c_2 M g(N) , \\ \frac{dM}{dt} &= -a_M M + c_2 M g(N) ,\end{aligned}$$

Kinetic proofreading: last exercise

Michaelis Menten:

$$F + L \xrightleftharpoons[k_{-1}]{k_1} C \quad \text{or} \quad \frac{dC}{dt} = k_1 FL - k_{-1}C \quad \text{with} \quad F = R - C \quad \text{gives} \quad C = \frac{RL}{K_m + L}$$

Kinetic proofreading:



$$\frac{dC_0}{dt} = k_1 FL - (k_{-1} + k_2)C_0, \quad \frac{dC_i}{dt} = k_2 C_{i-1} - (k_{-1} + k_2)C_i \quad \text{and} \quad \frac{dC_n}{dt} = k_2 C_{n-1} - k_{-1}C_n$$

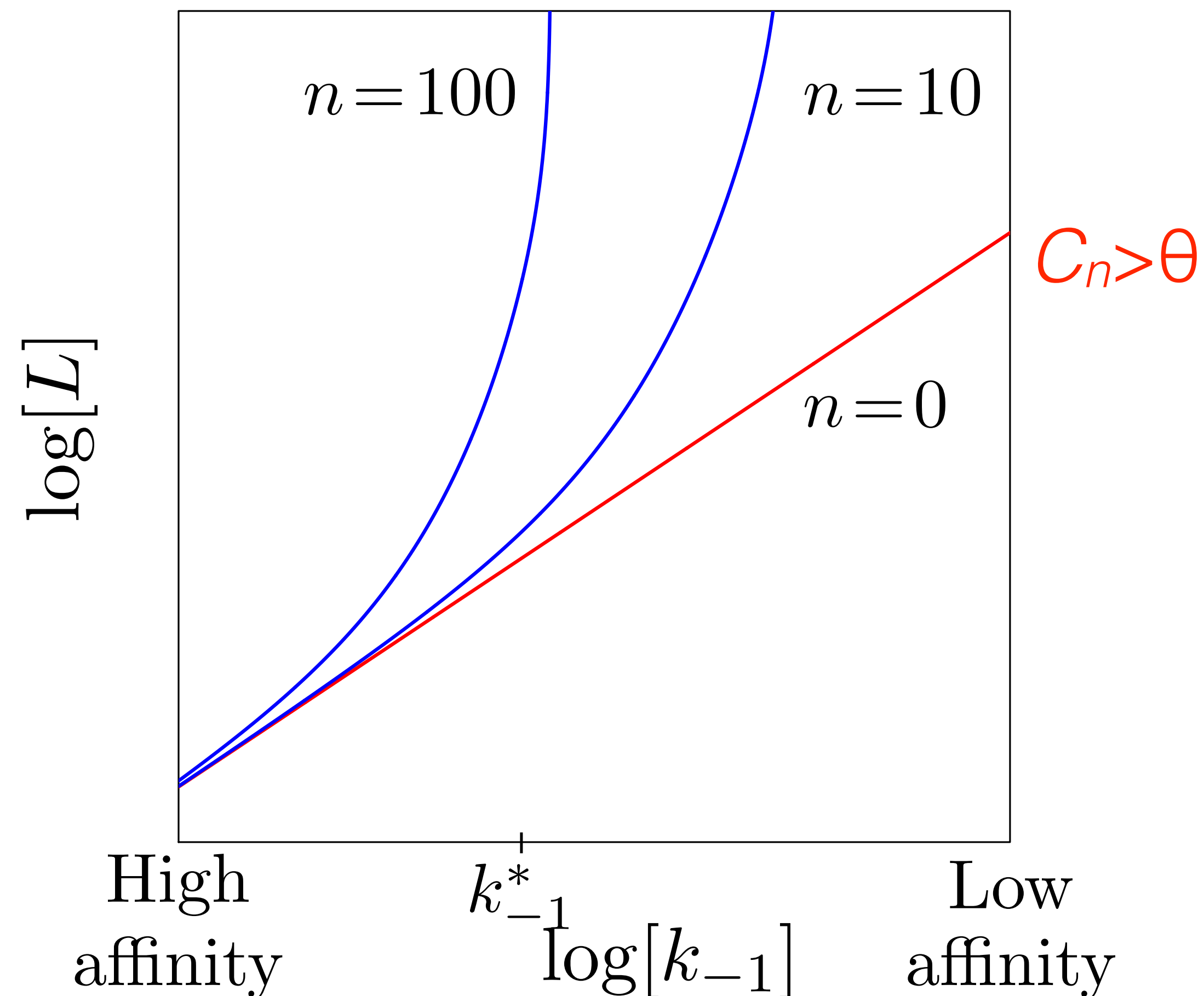
$$\text{with } F = R - \sum_i^n C_i \quad \text{gives} \quad \bar{C}_n = \frac{RL}{K_m + L} \left(\frac{k_2}{k_{-1} + k_2} \right)^n$$

When $k_{-1} \ll k_2$ similar to Michaelis Menten, otherwise discrimination

Kinetic proofreading: last exercise



$$\frac{dC_0}{dt} = k_1 F L - (k_{-1} + k_2) C_0, \quad \frac{dC_i}{dt} = k_2 C_{i-1} - (k_{-1} + k_2) C_i \quad \text{and} \quad \frac{dC_n}{dt} = k_2 C_{n-1} - k_{-1} C_n$$



$$\bar{C}_n = \frac{RL}{K_m + L} \left(\frac{k_2}{k_{-1} + k_2} \right)^n$$