

Chapter 3: Linear differential equations

The solution of $dx(t)/dt = ax(t)$ is $x(t) = Ce^{at}$, where $C = x(0)$.

Check this:

$$\partial_t Ce^{at} = aCe^{at} = ax(t)$$

Now two-dimensional systems:

$$\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y) \end{cases}$$

where $x(t)$ and $y(t)$ are unknown functions of time t , and f and g are functions of x and y .

An example:

$$\begin{cases} dx/dt = ax + by \\ dy/dt = cx + dy \end{cases} \quad \text{and} \quad \begin{cases} dx/dt = -2x + y \\ dy/dt = x - 2y \end{cases}$$

where x and y decay at a rate -1 , and are converted into one another at a rate 1 .

In matrix notation:

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We claim that $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ has as a general solution:

$$\begin{aligned} x(t) &= C_1 x_1 e^{\lambda_1 t} + C_2 x_2 e^{\lambda_2 t} \\ y(t) &= C_1 y_1 e^{\lambda_1 t} + C_2 y_2 e^{\lambda_2 t} \end{aligned}$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} e^{\lambda_2 t}$$

where $\lambda_{1,2}$ are eigenvalues and $(x_i \ y_i)$ are the corresponding eigenvectors of the matrix given above.

Like $x(t) = Ce^{at}$, this has only one steady state: $(x, y) = (0, 0)$.

Notice that the solutions $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} e^{\lambda_2 t}$ are a linear combination of the growth along the eigenvectors.

Since $x(t)$ and $y(t)$ grow when $\lambda_{1,2} > 0$ we obtain:

- a stable node when both $\lambda_{1,2} < 0$
- an unstable node when both $\lambda_{1,2} > 0$
- an (unstable) saddle point when $\lambda_1 > 0$ and $\lambda_2 < 0$ (or vice versa)

When $\lambda_{1,2}$ are complex, i.e., $\lambda_{1,2} = \alpha \pm i\beta$, we obtain

- a stable spiral when the real part $\alpha < 0$
- an unstable spiral when the real part $\alpha > 0$
- a neutrally stable center point when the real part $\alpha = 0$

Example: $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Since $\text{tr} = -4$ and $\det = 4 - 1 = 3$ we obtain:

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 - 12}}{2} = -2 \pm 1$$

so $\lambda_1 = -1$ and $\lambda_2 = -3$.

Hence solutions tend to zero and $(x, y) = (0, 0)$ is a stable node.

To find the eigenvector $\vec{v}_1$ we write:

$$\vec{v}_1 = \begin{pmatrix} -b \\ a - \lambda_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{or} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\vec{v}_2$ we write

$$\vec{v}_2 = \begin{pmatrix} -b \\ a - \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

In combination this gives

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

or

$$\begin{aligned} x(t) &= C_1 e^{-t} - C_2 e^{-3t} \\ y(t) &= C_1 e^{-t} + C_2 e^{-3t} \end{aligned}$$

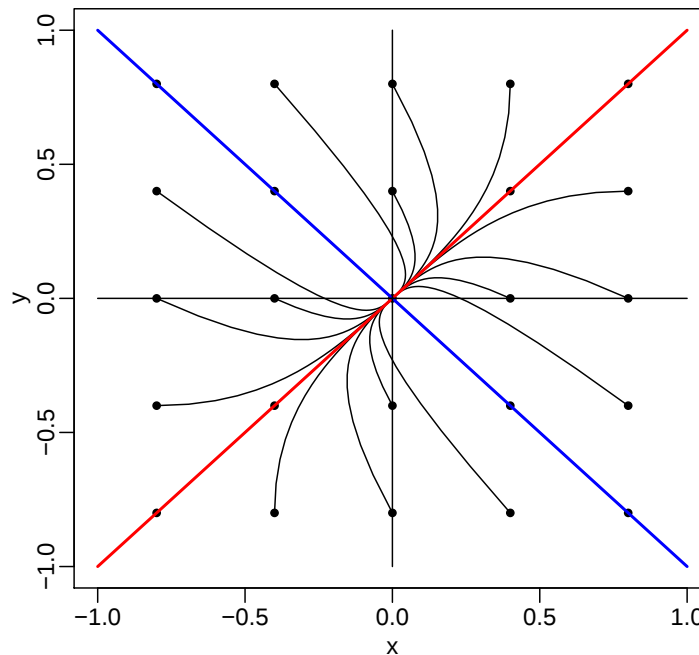
The integration constants C_1 and C_2 can be solved from the initial condition: i.e., $x(0) = C_1 - C_2$ and $y(0) = C_1 + C_2$.

Starting at $x(0) = y(0) = 1$, i.e., on the first eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$x(0) = 1 = C_1 - C_2 \quad \text{and} \quad y(0) = 1 = C_1 + C_2$$

we obtain $2 = 2C_1$ and find that $C_1 = 1$ and $C_2 = 0$. This means that this trajectory stays on the eigenvector obeying

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

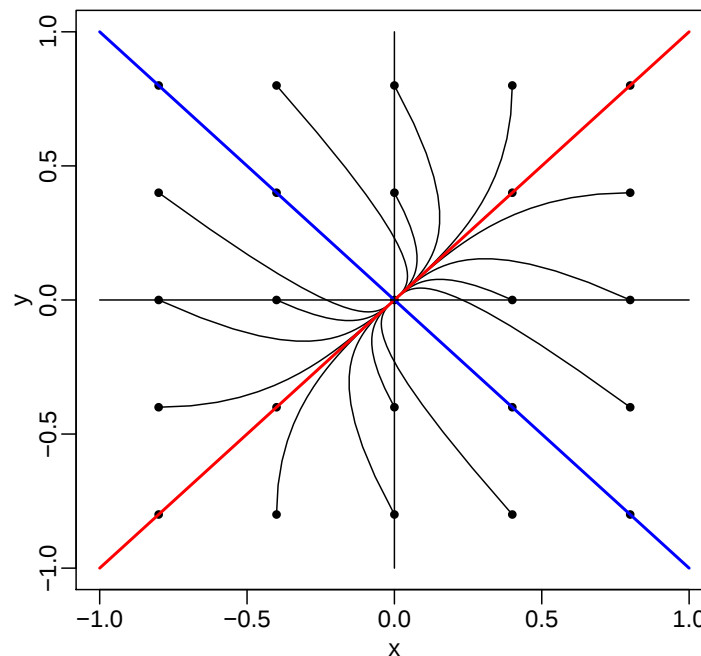


Starting at $(x(0), y(0)) = (1, 0.5)$, we solve from

$$x(0) = 1 = C_1 - C_2 \quad \text{and} \quad y(0) = 0.5 = C_1 + C_2$$

that $1.5 = 2C_1$, and hence $C_1 = 0.75$, $C_2 = -0.25$, and

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 0.75 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} - 0.25 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$



Chapter 4: Linear approximations

Derivative:

$$f'(\bar{x}) = \lim_{x \rightarrow \bar{x}} \frac{f(x) - f(\bar{x})}{x - \bar{x}} \quad \text{or} \quad f'(\bar{x}) = \lim_{h \rightarrow 0} \frac{f(\bar{x} + h) - f(\bar{x})}{h} ,$$

Rewrite this into:

$$f(x) \simeq f(\bar{x}) + f'(\bar{x}) (x - \bar{x}) \quad \text{or} \quad f(x) \simeq f(\bar{x}) + f'(\bar{x}) h ,$$

Example:

$$f(x) = ax^2 + b \quad \rightarrow \quad \partial_x f(x) = 2ax$$

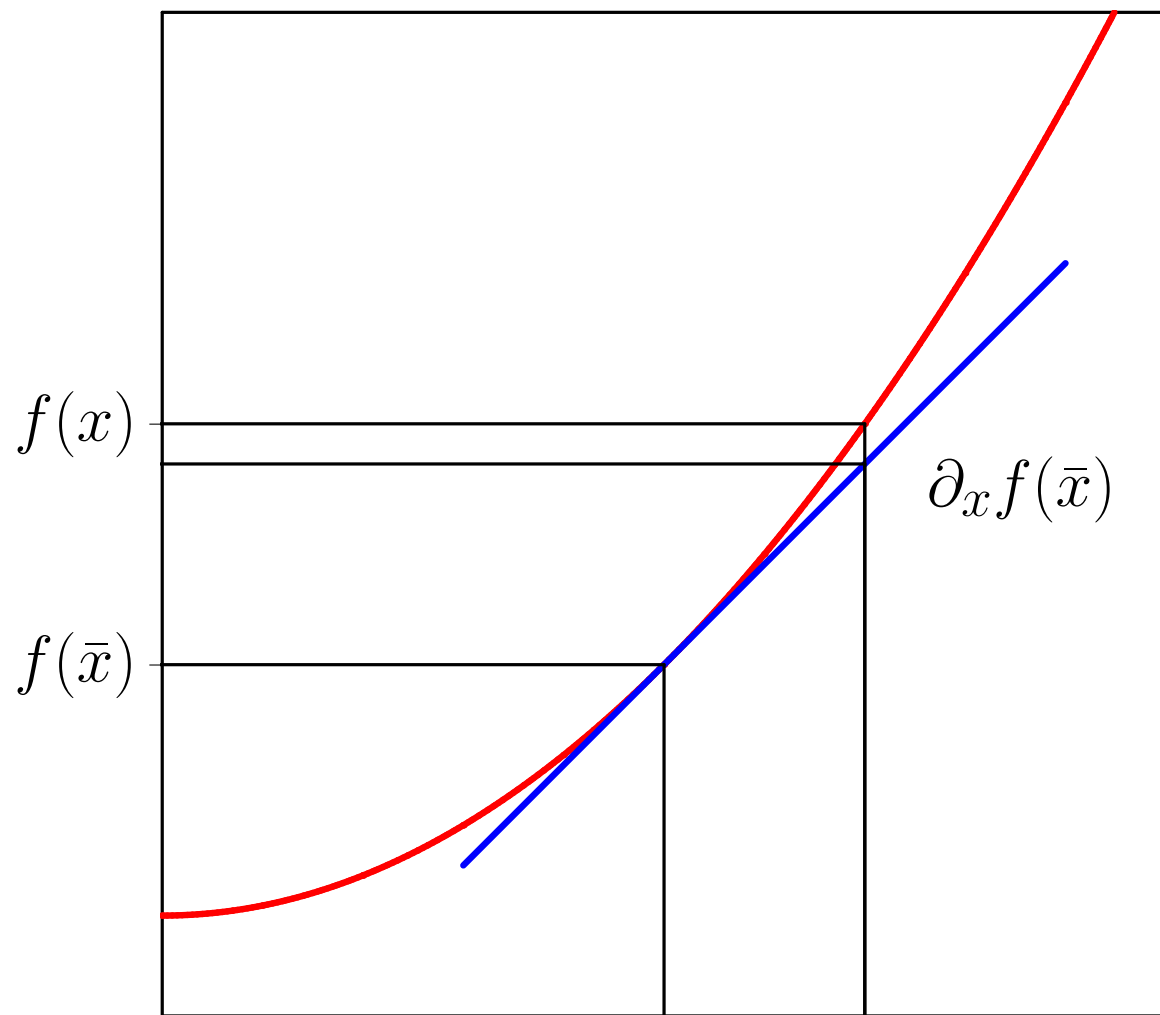
for $a = 2$, $b = 1$, $x = 3$ gives

$$f(3) = 2 \times 9 + 1 = 19 \quad \text{and} \quad \partial_x f(3) = 2 \times 2 \times 3 = 12$$

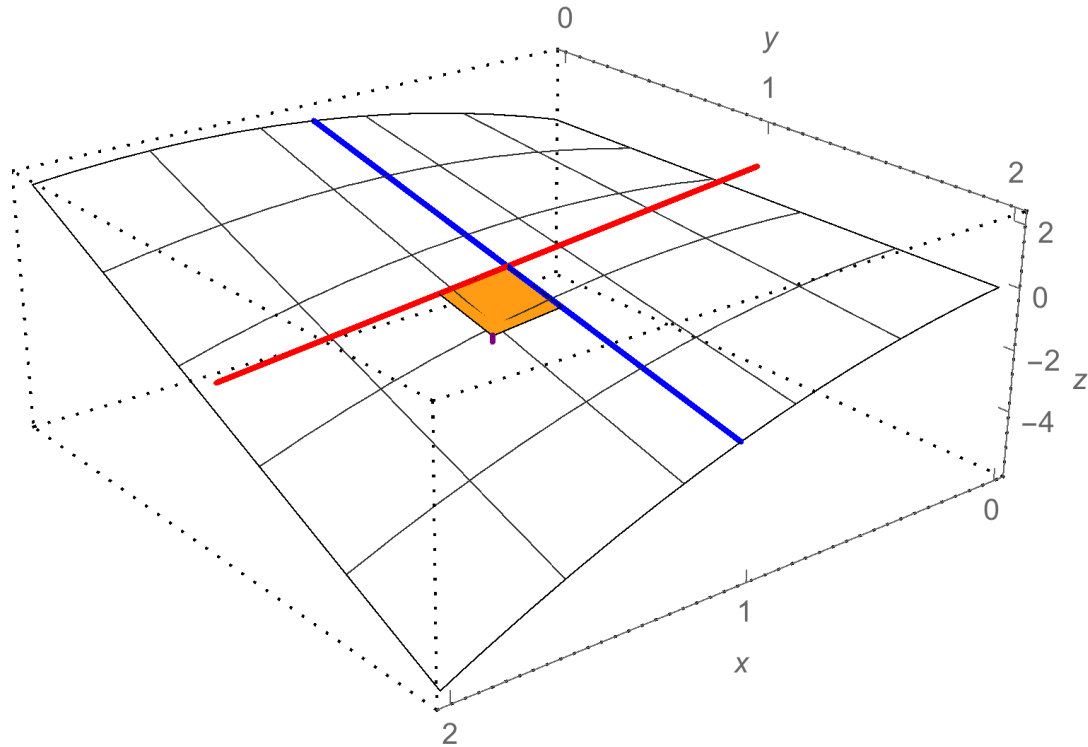
Hence

$$f(3.1) = 20.22 \quad \text{and} \quad f(3.1) \simeq f(3) + \partial_x f(3) \times 0.1 = 19 + 12 \times 0.1 = 20.2$$

$$f(x) \simeq f(\bar{x}) + \partial_x f(\bar{x}) (x - \bar{x})$$



The function $f(x, y) = 3x - x^2 - 2xy$:



$$\partial_x f(x, y) = 3 - 2x - 2y \quad \text{and} \quad \partial_y f(x, y) = -2x$$

and in the point $f(1, 1) = 0$:

$$\partial_x f(x, y) = -1 \quad \text{and} \quad \partial_y f(x, y) = -2$$

Generally

$$f(x, y) \simeq f(\bar{x}, \bar{y}) + \partial_x f (x - \bar{x}) + \partial_y f (y - \bar{y})$$

Or, after defining $h_x = x - \bar{x}$ and $h_y = y - \bar{y}$:

$$f(x, y) = f(\bar{x} + h_x, \bar{y} + h_y) \simeq f(\bar{x}, \bar{y}) + \partial_x f h_x + \partial_y f h_y$$

Example:

$$f(x, y) = 3x - x^2 - 2xy, \quad f(1, 1) = 0, \quad \partial_x = -1, \partial_y = -2$$

$$f(1.25, 1.25) = 3.75 - 1.5625 - 3.125 = -0.9375$$

$$f(1.25, 1.25) \simeq 0 - 1 \times 0.25 - 2 \times 0.25 = -0.75$$

Consider

$$\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y) \end{cases}$$

close to an equilibrium point at $(\bar{x}, \bar{y})$, i.e., $f(\bar{x}, \bar{y}) = g(\bar{x}, \bar{y}) = 0$

Linear approximation of $f(x, y)$ close to the equilibrium:

$$f(x, y) \simeq f(\bar{x}, \bar{y}) + \partial_x f (x - \bar{x}) + \partial_y f (y - \bar{y})$$

As $f(\bar{x}, \bar{y}) = 0$ we obtain

$$f(x, y) \simeq \partial_x f (x - \bar{x}) + \partial_y f (y - \bar{y})$$

For $g(x, y)$ this yields:

$$g(x, y) \simeq \partial_x g (x - \bar{x}) + \partial_y g (y - \bar{y})$$

$$\begin{cases} dx/dt = f(x, y) \\ dy/dt = g(x, y) \end{cases} \quad \text{became} \quad \begin{cases} dx/dt \simeq \partial_x f (x - \bar{x}) + \partial_y f (y - \bar{y}) \\ dy/dt \simeq \partial_x g (x - \bar{x}) + \partial_y g (y - \bar{y}) \end{cases}$$

Since the partial derivatives are merely the slopes of $f(x, y)$ and $g(x, y)$ at the point $(\bar{x}, \bar{y})$, they are constants that we can write as

$$a = \partial_x f, \quad b = \partial_y f, \quad c = \partial_x g, \quad d = \partial_y g$$

Steady states $\bar{x}$ and $\bar{y}$ are also constants, with derivatives zero:

$$\frac{dx}{dt} = \frac{dx}{dt} - \frac{d\bar{x}}{dt} = \frac{d(x - \bar{x})}{dt} \quad \text{and} \quad \frac{dy}{dt} = \frac{dy}{dt} - \frac{d\bar{y}}{dt} = \frac{d(y - \bar{y})}{dt}$$

Hence

$$\begin{cases} d(x - \bar{x})/dt = a(x - \bar{x}) + b(y - \bar{y}) \\ d(y - \bar{y})/dt = c(x - \bar{x}) + d(y - \bar{y}) \end{cases}$$

Changing variables to the distances $h_x = x - \bar{x}$ and $h_y = y - \bar{y}$:

$$\begin{cases} dh_x/dt = ah_x + bh_y \\ dh_y/dt = ch_x + dh_y \end{cases}$$

having the solution

$$\begin{pmatrix} h_x(t) \\ h_y(t) \end{pmatrix} = C_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} e^{\lambda_2 t}$$

where $\lambda_{1,2}$ and $(x_i \ y_i)$ are the eigenvalues and corresponding eigenvectors of the Jacobi matrix

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Knowing the two eigenvalues of

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the steady state will be stable when $\lambda_1 < 0$ and $\lambda_2 < 0$.

If so the return time is defined by the largest eigenvalue:

$$T_R = \frac{-1}{\max(\lambda_1, \lambda_2)}$$

Example:

$$\frac{dx}{dt} = f(x, y) = a - bx - cxy \quad \text{and} \quad \frac{dy}{dt} = g(x, y) = dxy - ey ,$$

with $\bar{x} = \frac{a}{b}$ when $y = 0$, and $\bar{x} = \frac{e}{d}$ and $\bar{y} = \frac{ad}{ce} - \frac{b}{c}$

The Jacobian is:

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix}$$

Into the Jacobian

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix}$$

we first fill in $\bar{x} = \frac{a}{b}$ and $\bar{y} = 0$,

$$J = \begin{pmatrix} -b & -\frac{ca}{b} \\ 0 & \frac{da}{b} - e \end{pmatrix}$$

Since this matrix is in a diagonal form we know that the diagonal elements provide the eigenvalues, i.e., $\lambda_1 = -b$ and $\lambda_2 = \frac{da}{b} - e$.

Hence this state is stable whenever $\lambda_2 < 0$, i.e., $\frac{a}{b} < \frac{e}{d}$.

For the same Jacobian

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix}$$

now consider $\bar{x} = \frac{e}{d}$ and $\bar{y} = \frac{ad}{ce} - \frac{b}{c}$, and first fill in $\bar{x}$:

$$J = \begin{pmatrix} -b - c\bar{y} & -\frac{ce}{d} \\ d\bar{y} & 0 \end{pmatrix}$$

When $\bar{y} > 0$ the signs of this matrix are given by

$$J = \begin{pmatrix} -\alpha & -\beta \\ \gamma & 0 \end{pmatrix} \quad \text{with} \quad \text{tr} J = -\alpha < 0 \quad \text{and} \quad \det J = \beta\gamma > 0 ,$$

such that

$$\lambda_{1,2} = \frac{\text{tr} \pm \sqrt{\text{tr}^2 - 4 \det}}{2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta\gamma}}{2} < 0 ,$$

Since $\lambda_{1,2} < 0$ the non-trivial steady state is stable.

Numerical example: $a = b = c = d = 1$ and $e = 0.5$ in the same

$$\frac{dx}{dt} = f(x, y) = a - bx - cxy \quad \text{and} \quad \frac{dy}{dt} = g(x, y) = dxy - ey ,$$

mean that $\bar{x} = 1$ when $y = 0$, and $\bar{x} = 0.5$ and $\bar{y} = 1$, otherwise.

The Jacobian of the trivial steady state $(\bar{x}, \bar{y}) = (1, 0)$ is

$$J = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.5 \end{pmatrix} ,$$

meaning that $\lambda_1 = -1$ and $\lambda_2 = 0.5$: this is a saddle-point.

The Jacobian of the non-trivial steady state $(\bar{x}, \bar{y}) = (0.5, 1)$ is

$$J = \begin{pmatrix} -b - c\bar{y} & -c\bar{x} \\ d\bar{y} & d\bar{x} - e \end{pmatrix} = \begin{pmatrix} -2 & -0.5 \\ 1 & 0 \end{pmatrix}$$

with $\text{tr} J = -2$ and $\det J = 0.5$.

The Jacobian of the non-trivial steady state

$$J = \begin{pmatrix} -2 & -0.5 \\ 1 & 0 \end{pmatrix} ,$$

with $\text{tr} J = -2$ and $\det J_2 = 0.5$ has eigenvalues

$$\lambda_{1,2} = \frac{\text{tr} \pm \sqrt{\text{tr}^2 - 4 \det}}{2} = \frac{-2 \pm \sqrt{4 - 2}}{2}$$

giving

$$\lambda_1 = -1 + \frac{\sqrt{2}}{2} \simeq -0.29 \quad \text{and} \quad \lambda_2 = -1 - \frac{\sqrt{2}}{2} \simeq -1.71$$

revealing that this is a stable node.

Chapter 5: Efficient analysis

Having

$$J = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we know that

$$\lambda_{1,2} = \frac{\text{tr} \pm \sqrt{D}}{2} \quad \text{where} \quad D = \text{tr}^2 - 4 \det \quad \text{is the discriminant}$$

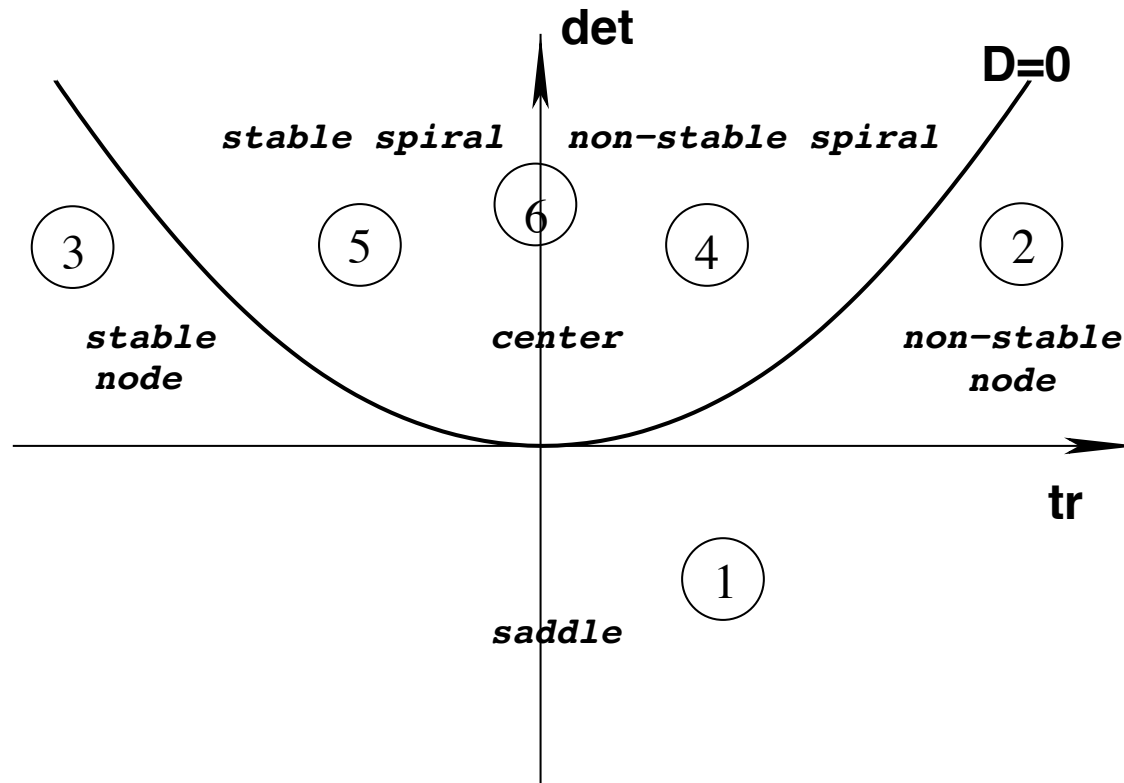
Observing that

$$\lambda_1 + \lambda_2 = \text{tr}[J] \quad \text{and} \quad \lambda_1 \times \lambda_2 = \det[J] ,$$

the latter because

$$\frac{1}{4}(\text{tr} + \sqrt{D})(\text{tr} - \sqrt{D}) = \frac{1}{4}(\text{tr}^2 - D) = \frac{1}{4}(\text{tr}^2 - \text{tr}^2 + 4 \det) = \det$$

we can classify steady states by just the trace and determinant of their Jacobi matrix.



$$\lambda_{1,2} = \frac{\text{tr} \pm \sqrt{D}}{2}$$

$$D = \text{tr}^2 - 4 \det$$

$$\lambda_1 + \lambda_2 = \text{tr}$$

$$\lambda_1 \times \lambda_2 = \det$$

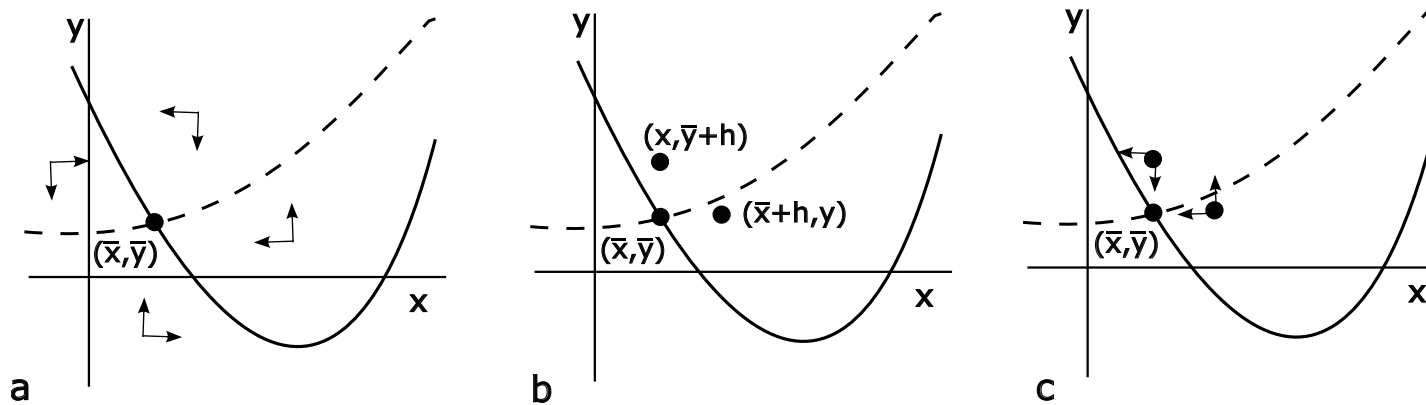
1. if $\det < 0$ then $D > 0$: $\lambda_{1,2}$ are real with unequal sign: saddle
2. if $\det > 0$, $\text{tr} > 0$ and $D > 0$ then $\lambda_{1,2} > 0$: unstable node.
3. if $\det > 0$, $\text{tr} < 0$ and $D > 0$ then $\lambda_{1,2} < 0$: stable node.
4. if $\det > 0$, $\text{tr} > 0$ and $D < 0$ then $\lambda_{1,2} > 0$: unstable spiral.
5. if $\det > 0$, $\text{tr} < 0$ and $D < 0$ then $\lambda_{1,2} > 0$: stable spiral.

Graphical Jacobian: use the signs only

$$J = \begin{pmatrix} \partial_x f \simeq \frac{f(\bar{x} + h, \bar{y})}{h} & \partial_y f \simeq \frac{f(\bar{x}, \bar{y} + h)}{h} \\ \partial_x g \simeq \frac{g(\bar{x} + h, \bar{y})}{h} & \partial_y g \simeq \frac{g(\bar{x}, \bar{y} + h)}{h} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\text{tr}[J] = \alpha + \delta$ and $\det[J] = \alpha\delta - \beta\gamma$.

If $\text{tr} < 0$ and $\det > 0$ the state will be stable.



$$J = \begin{pmatrix} \partial_x f \simeq \frac{f(\bar{x} + h, \bar{y})}{h} & \partial_y f \simeq \frac{f(\bar{x}, \bar{y} + h)}{h} \\ \partial_x g \simeq \frac{g(\bar{x} + h, \bar{y})}{h} & \partial_y g \simeq \frac{g(\bar{x}, \bar{y} + h)}{h} \end{pmatrix} = \begin{pmatrix} - & - \\ + & - \end{pmatrix}$$

Hence the point is stable because $\text{tr} = -\alpha - \delta < 0$ and

$$\det = (-\alpha \times -\delta) - (-\beta \times \gamma) = \alpha\delta + \beta\gamma > 0$$